RG fixed point tensors and factorisation in 2D CFT

and ~~holographic tensor networks~~

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ExU YITP Holography, Gravity and Quantum Information

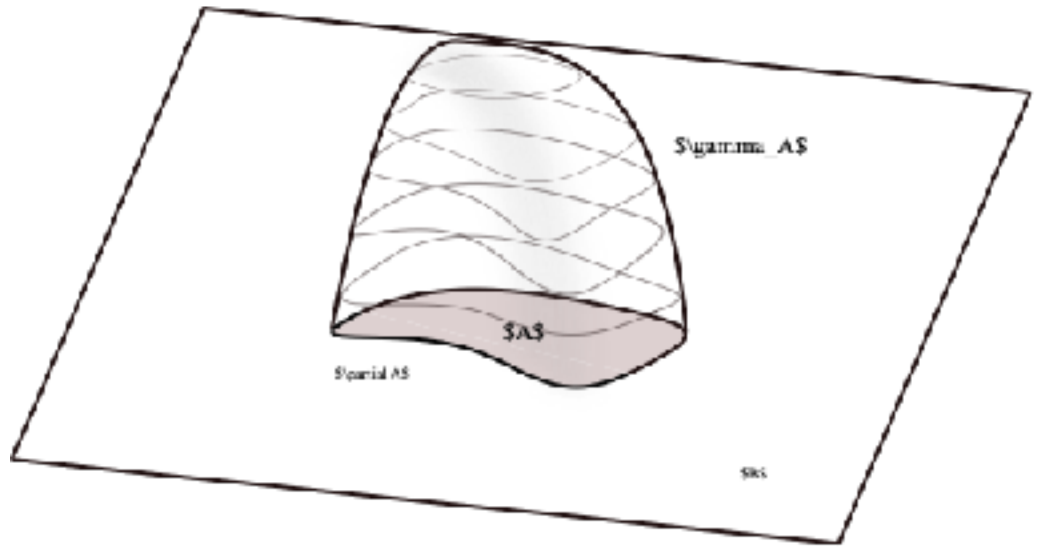
Work done in collaboration with :
Lin Chen, Ruoshui Wang, Haochen Zhang, Kaixin Ji, Xiangdong Zeng,
Exact Holographic Networks From Topological Orders arXiv:2210.12127
+ Lin Chen, Gong Cheng, Zheng-Cheng Gu, Yikun Jiang, Bingxin Lao (I, II to appear) + ongoing
+ A continuation of :
Arpan Bhattacharya, LYH, Yang Lei, Wei Li, Charles Melby-Thompson,
***JHEP* 04 (2019) 170, *JHEP* 05 (2019) 118, *JHEP* 01 (2018) 139, *JHEP* 08 (2016) 086**
Gabriel Wong, LYH ***Phys.Rev.D* 104 (2021) 2, 026012**
Lin Chen, Xirong Liu LYH
***Phys.Rev.Lett.* 127 (2021) 22, 221602, *JHEP* 09 (2021) 097 , *JHEP* 06 (2021) 094**

Yau Mathematical Sciences Center, Tsinghua University

Many body entanglement and holographic theories

AdS/CFT says entanglement is geometry

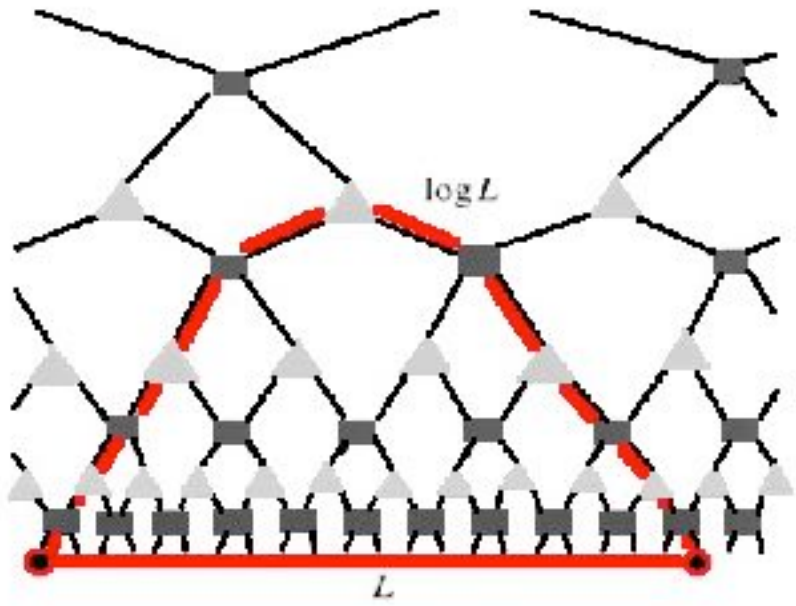
Ryu-Takayanagi Formula:
(2006)



$$S_{EE} = \frac{A}{4G}$$

Tensor network is a geometrization of entanglement. It is explicitly local.

Tensor network is a framework to construct models that realise these ideas



Picture courtesy Orus

Brian Swingle (2012)

Overview

- Fixed point of tensor renormalisation group (TRG) /tensor network renormalisation (TNR)

- CFT factorisation and entanglement brane boundary condition

- Tensor network from 3D TQFT , RG operator, Holographic Tensor Network and 2D fixed points

  Checking it numerically in Ising theory —
  a) shrinking condition
  b) recovering the closed string spectrum from the open correlation functions

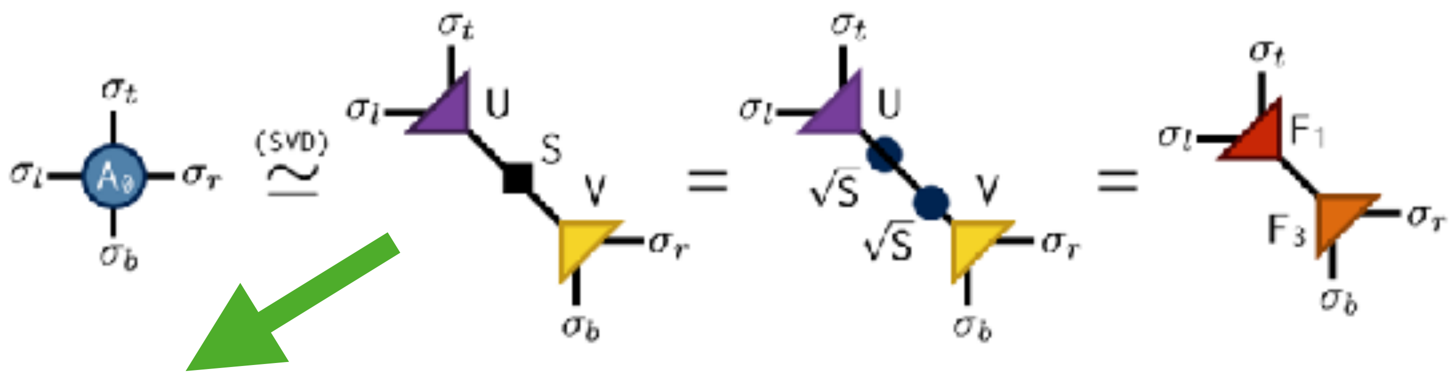
- Holographic Tensor network for Liouville Theory (?)

- Outlook

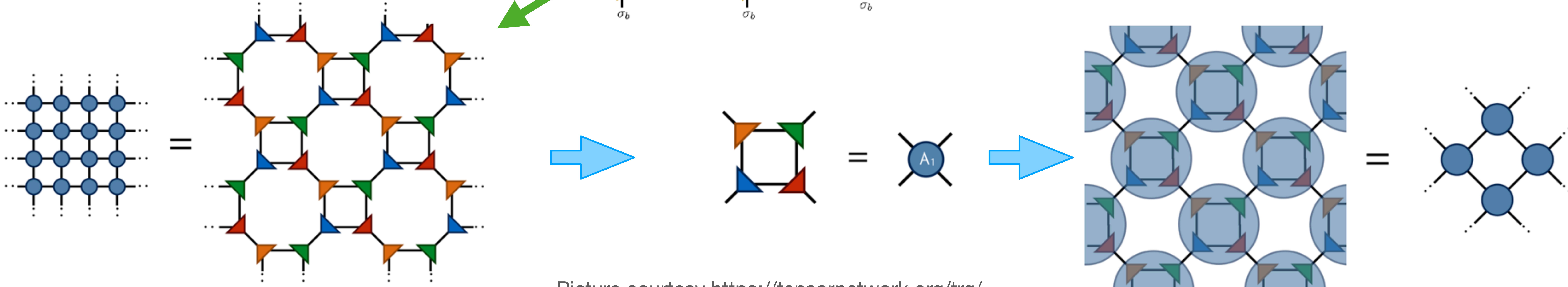
TRG/ TNR fixed point and CFT

Levin, Nave (2007); Gu, Levin, Wen (2008); Evenbly, Vidal (2015) ; Evenbly (2017); Yang, Gu, Wen (2017) ……..

- Many classical statistical models adopt tensor network representation (see Prof. Nishino's lecture last Friday and Prof. Meurice's talk yesterday)
- It is well know that at the critical point, there is a very efficient way of extracting CFT data. that is — performing TRG/TNR and look for the fixed point:



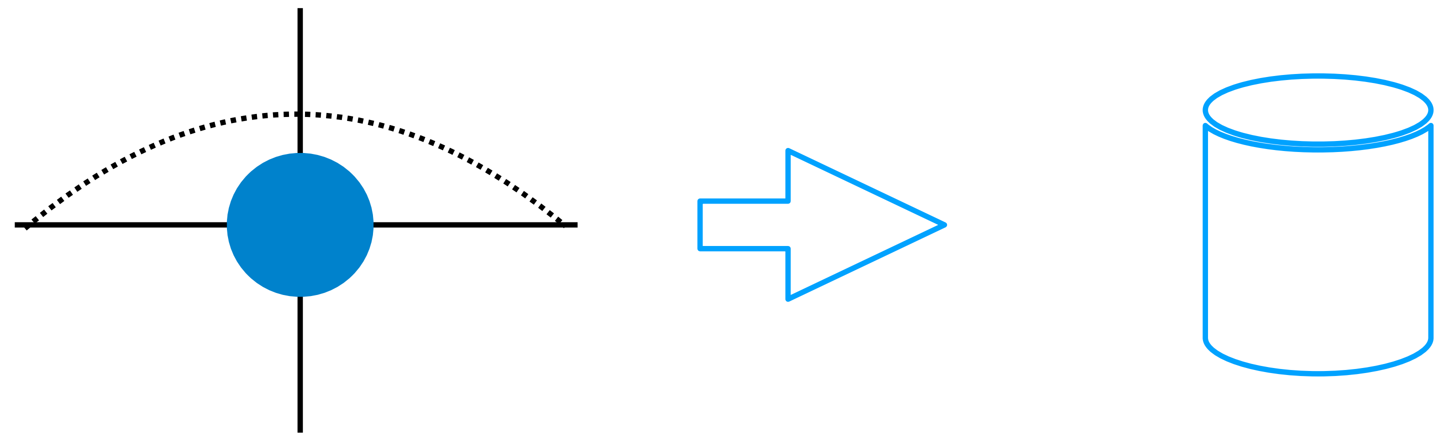
e.g. by SVD to decompose the tensor into products of two



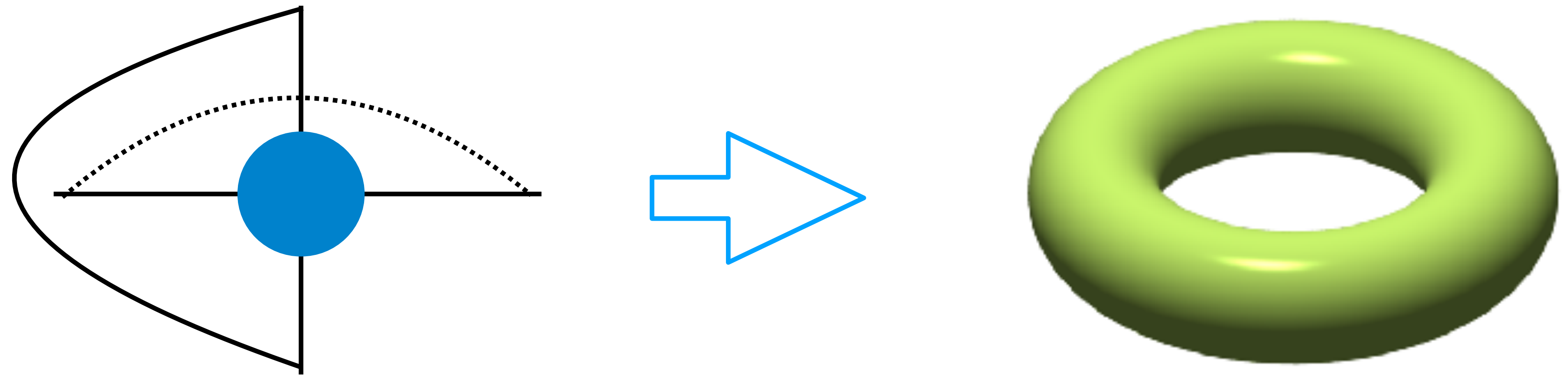
Picture courtesy https://tensornetwork.org/trg/

TRG/ TNR fixed point and CFT

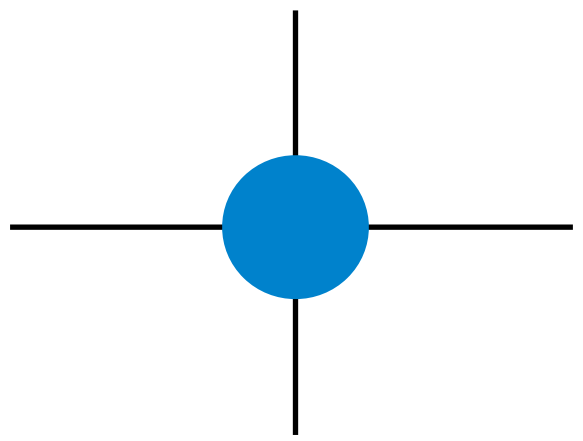
Levin, Nave (2007); Gu, Levin, Wen (2008); Evenbly, Vidal (2015) ; Evenbly (2017); Yang, Gu, Wen (2017) ……..



Solve for the eigenvalues of this spectrum gives the closed spectrum of the CFT in the thermodynamic limit



- We can extract CFT data from the fixed point tensor:
- In general the bond dimensions of each tensor is kept at finite. The recovery of spectrum only approximate — only the low lying states match well with the exact CFT data. With higher bond dimensions, more states in the spectrum recovered.
- There should be a CFT interpretation of the fixed point tensor

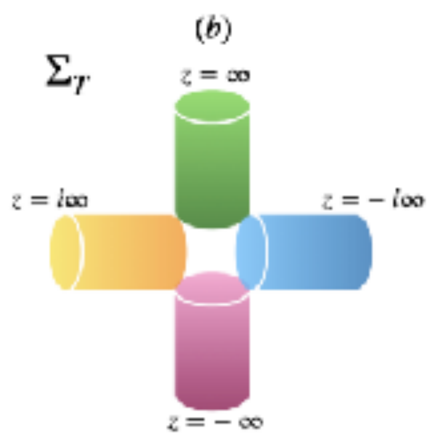


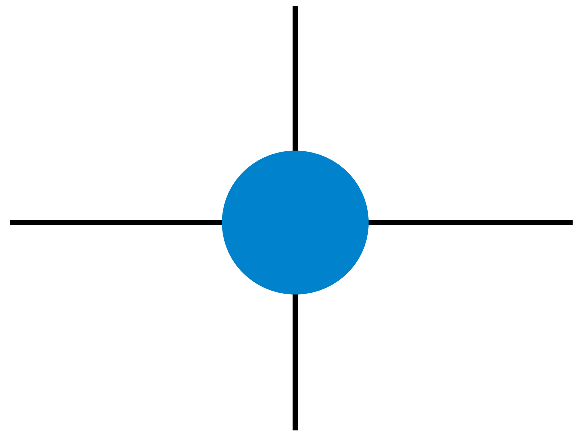
TRG/ TNR fixed point and CFT

The Closed correlation point of view:

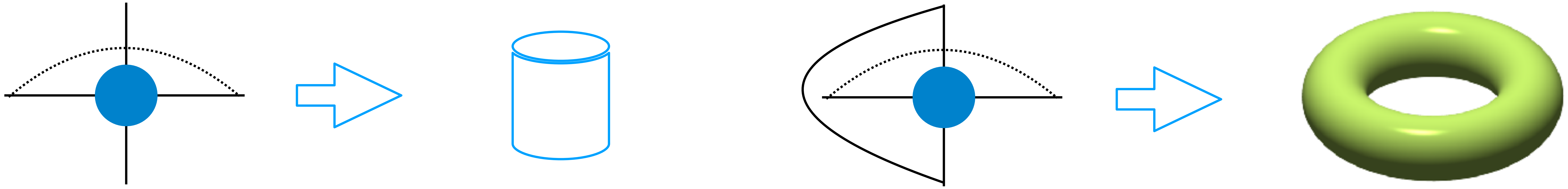
The Closed Picture: This tensor is related to closed string correlation functions?
Yang, Gu, Wen (2017) ; Ueda, Yamazaki (2023) arXiv: 2307.02523

picture courtesy 2307.02523



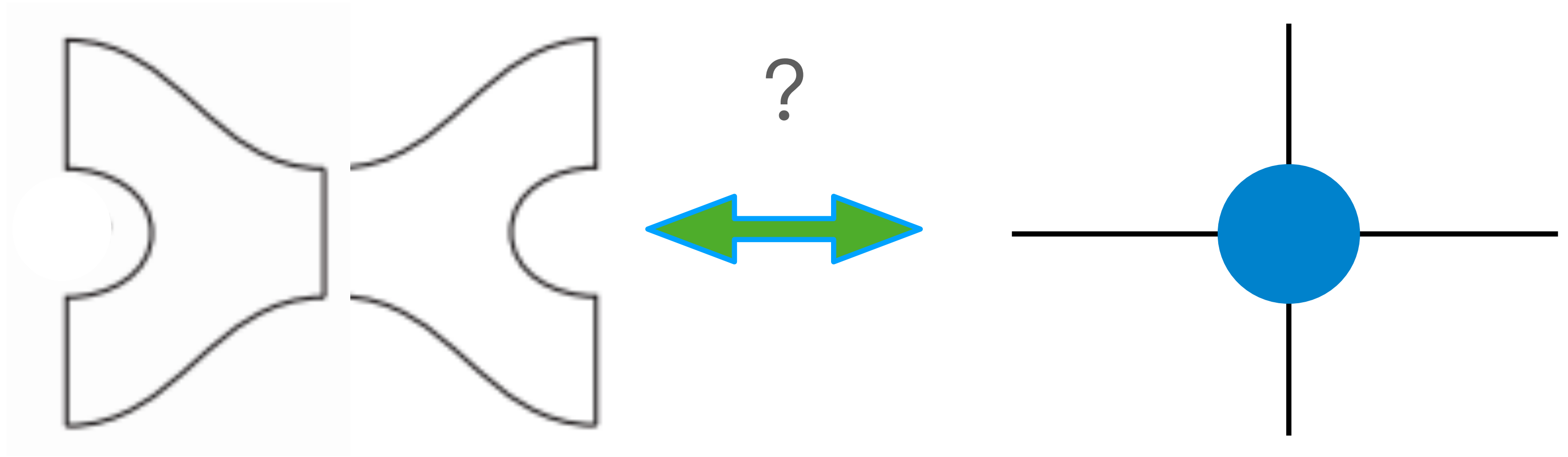


But it could not (yet) reproduce the following computations — I believe, for reasons that the topology does not quite work out…

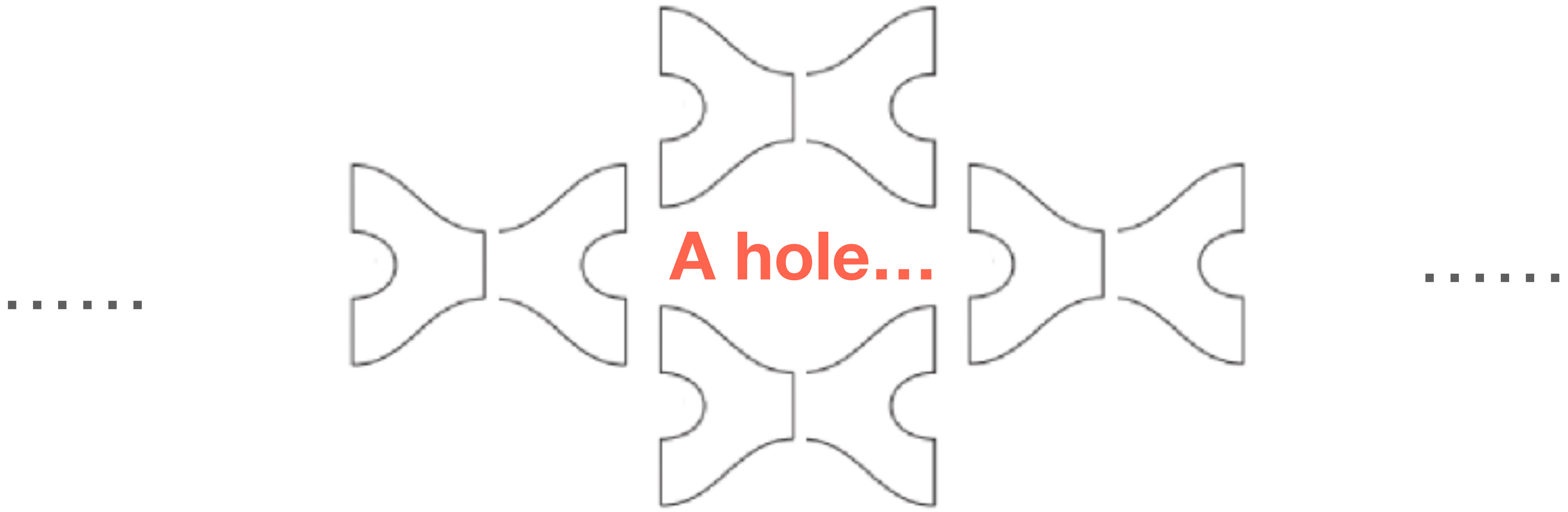


TRG/ TNR fixed point and CFT

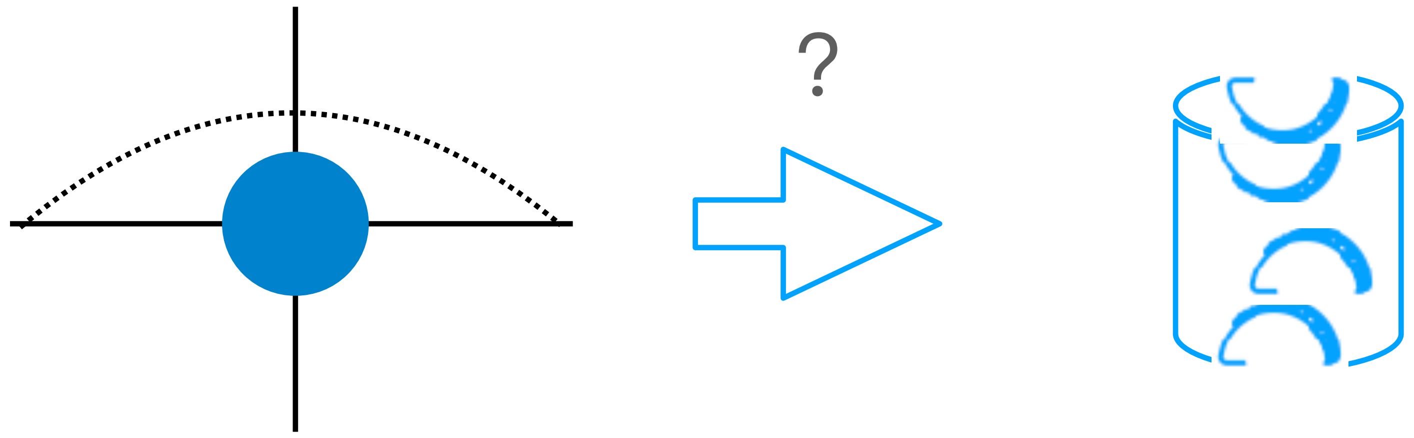
The open correlation point of view:

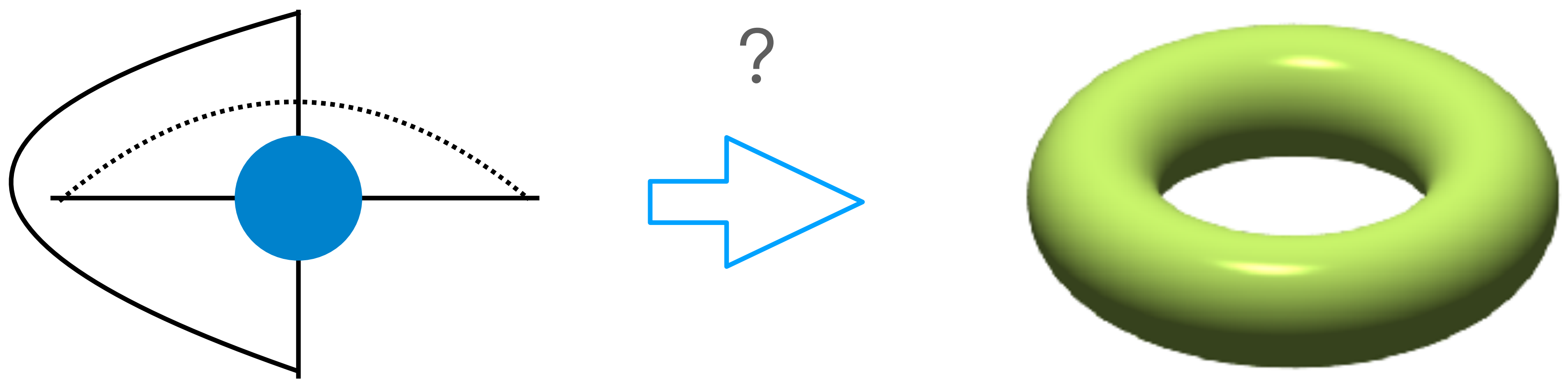


The open correlation function point of view:
let open correlation function tiles a planar surface —
Z. Gu, G. Cheng (2015/16 ? unpublished — the hole didn't close for fixed conformal boundary conditions. The spectrum on the cylinder was very far from the CFT results… gave up)



What do we do with holes ? We want them to shrink to nothingness—
this gives us the original closed string partition function.
This closing a hole business is related to this question of factorisation of CFT.
Shrinkable bc =
"Entanglement Brane Boundary Condition" in RCFT
G. Wong, LYH 2020



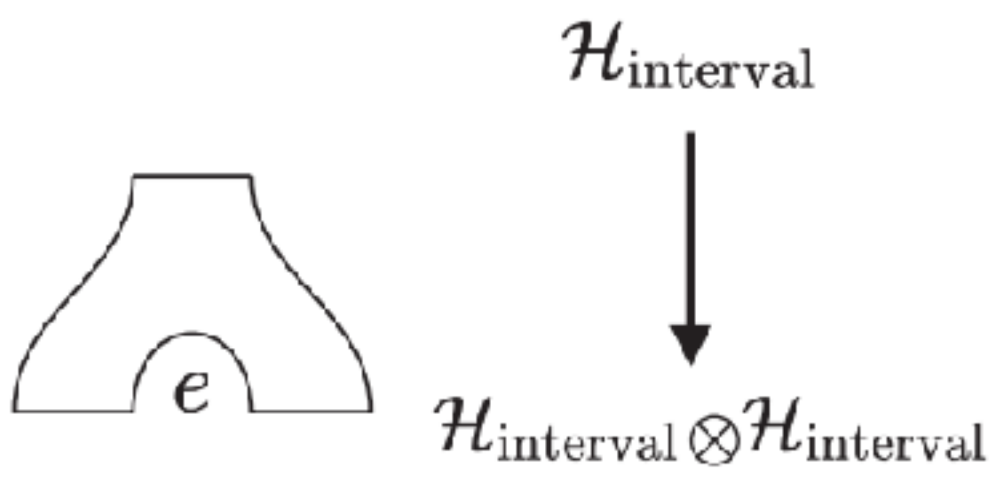


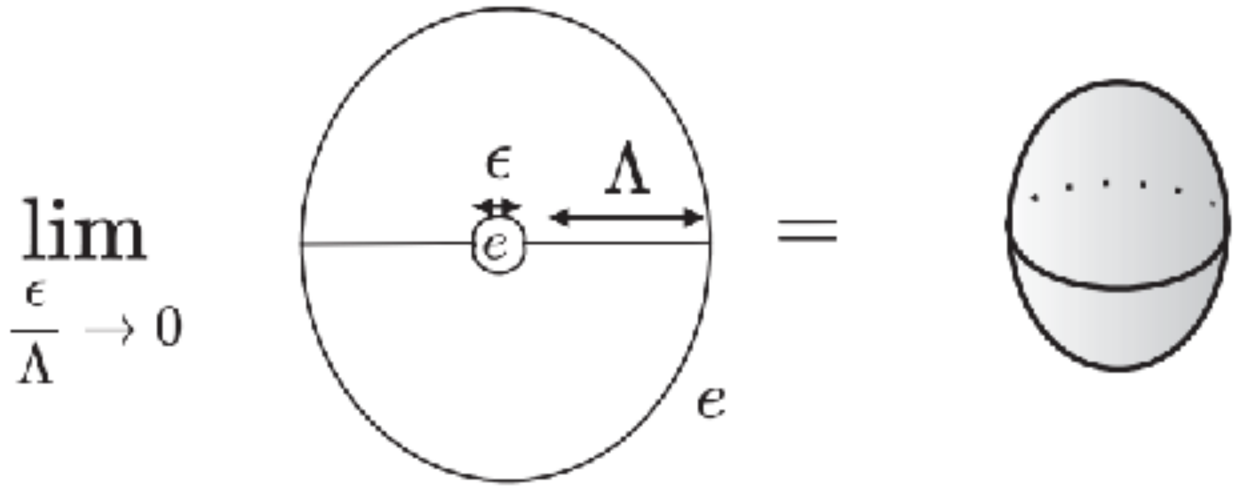
CFT Factorisation and Entanglement Brane Boundary Conditions

Wong, LYH 2020

If holes can close then the idea of Gu and Cheng might be on the right track ???

When computing entanglement entropy, one would have to introduce a boundary between the splitter intervals.





The condition one would naively thought should be imposed on this boundary is that when we take overlap of the states, the holes should close. We therefore propose that the entanglement brane boundary condition (in the closed string channel) to be the “0” Ishibashi state in a diagonal RCFT

Ishibashi state

$$|a\rangle = \sum_b S_{ba}/\sqrt{S_{b0}}\,|b\rangle\rangle$$

$$|e\rangle\rangle = \sum_a c_a |a\rangle$$

$$c_a = \sqrt{S_{00} S_{a0}}$$

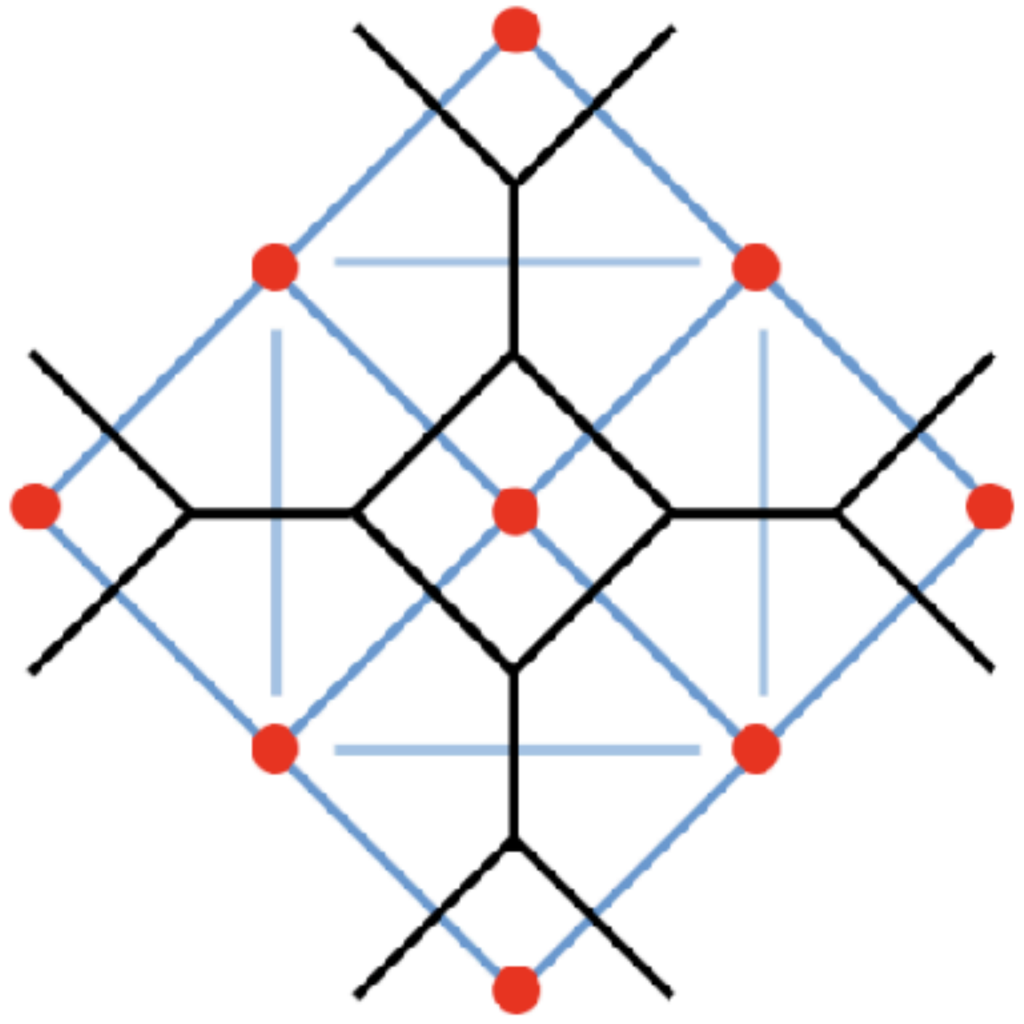
quantum dimension of sector a

$$S_{a0} = d_a / D$$

total quantum dimension

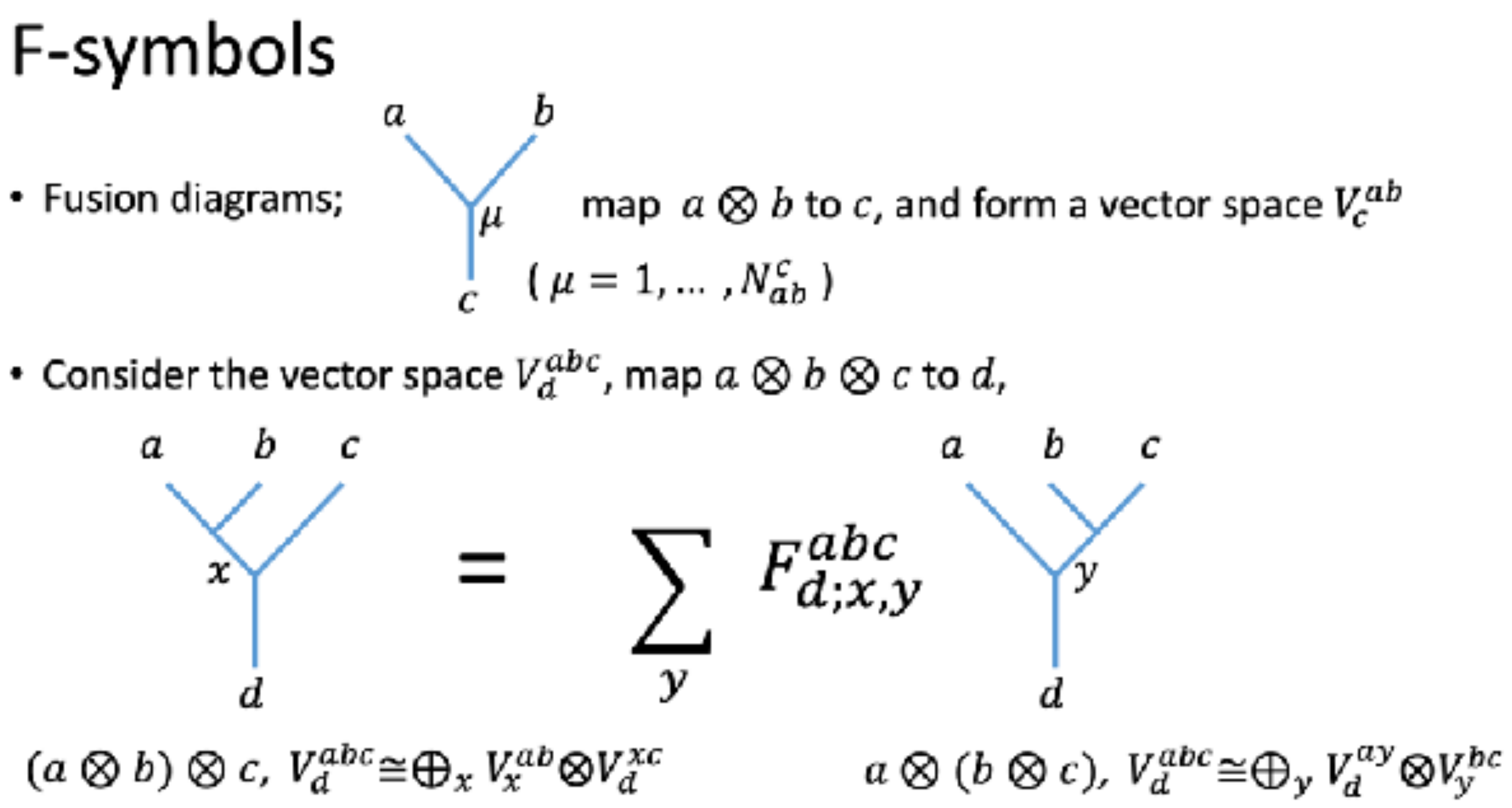
2D CFT and 3D TQFT

From integrable lattice models to fixed points of RG operator and holographic tensor network



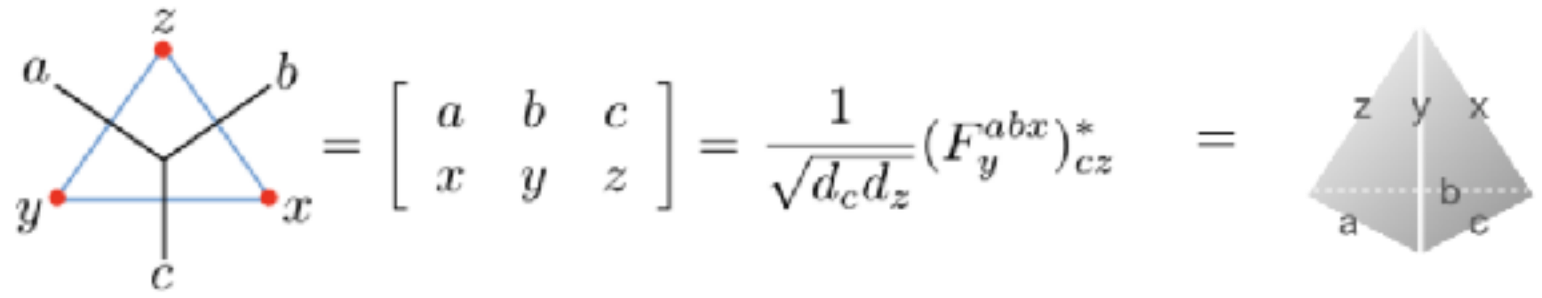
RSOS integrable models from “strange correlates”

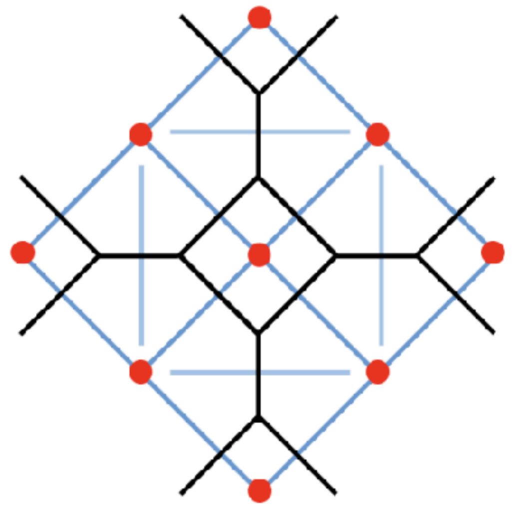
Feiguin, Trebst, Ludwig, Troyes, Kitaev, Wang, Freedman 2007;
Aasen, Fendley, Mong J. Phys. A; Math. Theor. 2016; 2020 ;
Bal, Williamson, Vanhove, Bultinck, Haegeman, Verstraete 2017;
Lootens, Vanhove, Verstraete 2019 ;



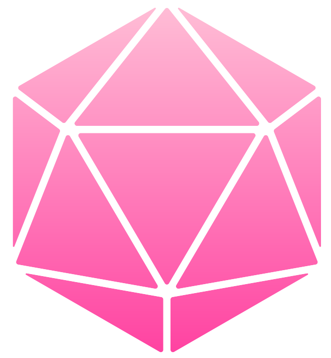
- An interesting observation:

Take the PEP tensor network representation of the ground state $|\Psi_a^{LW}\rangle$ of a Levin Wen model (= Turaev-Viro formulation of TQFT) defined on a time slice with some triangulation with lattice constant “a”

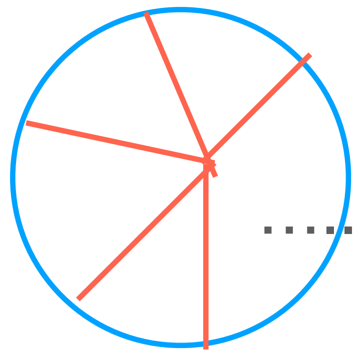




Radius of sphere goes to infinity



triangulation of a sphere into tetrahedrons



cross section of the triangulated ball

According to the Turaev-Viro formulation, red dots are interior edges have to be summed over, weighted by d_i

Gu, Levin, Swingle, Wen PRB 2009;
Buershaper, Aguado, Vidal PRB 2009;
(More recently — the form we follow closely , is presented in Bultinck, Marien, williamson, Sahinoglu, Haegeman, Verstraete Annals of physics 2017; Williamson, Bultinck, Verstraete 2017)

RSOS integrable models from “strange correlates”

Feiguin, Trebst, Ludwig, Troyes, Kitaev, Wang, Freedman 2007;
Aasen, Fendley, Mong J. Phys. A; Math. Theor. 2016; 2020 ;
Bal, Williamson, Vanhove, Bultinck, Haegeman, Verstraete 2017;
Lootens, Vanhove, Verstraete 2019 ;

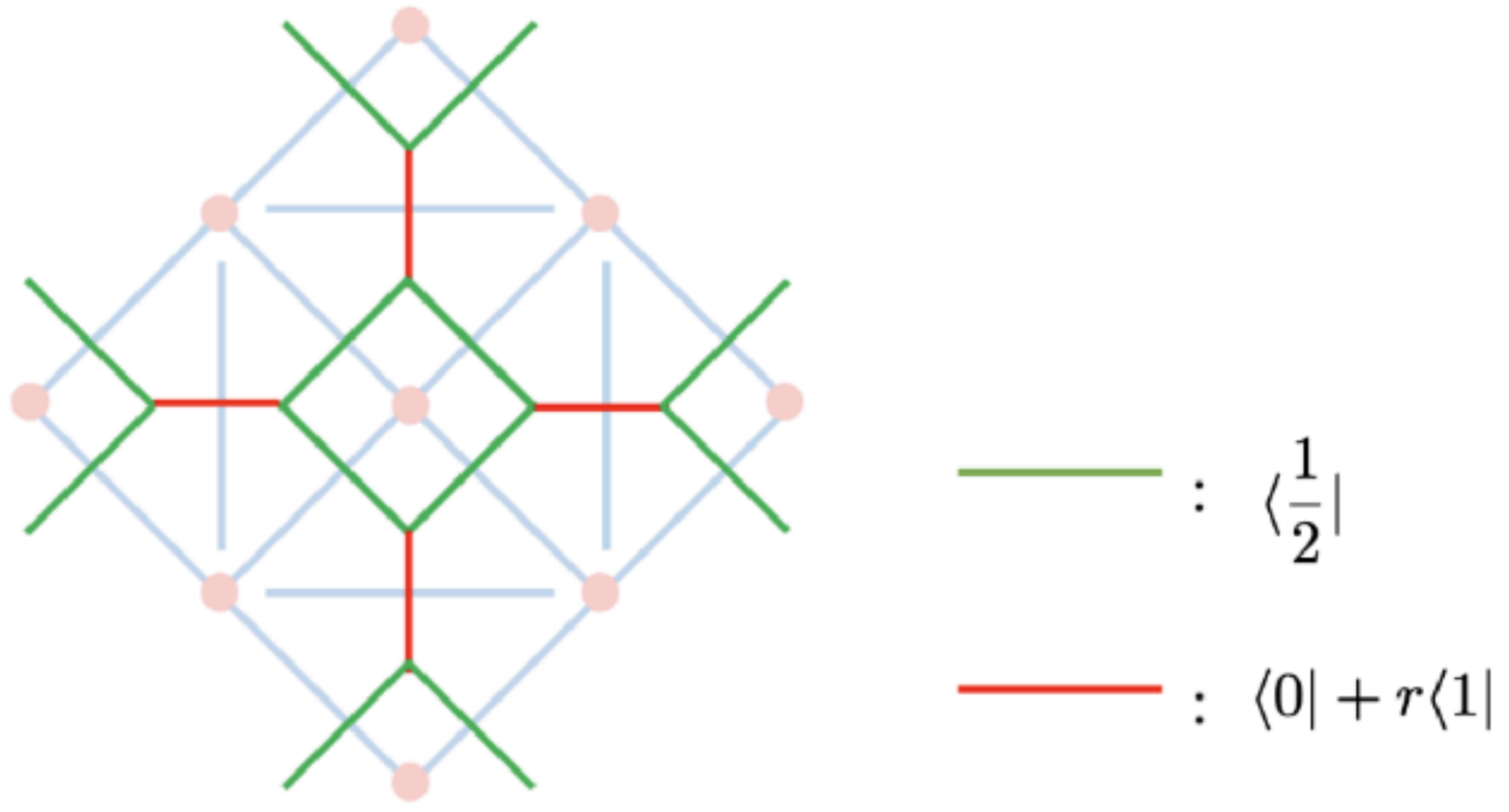
- An interesting observation:

Pick some mysterious direct product state $\langle \Omega_N |$ and take the overlap with $|\Psi_a^{LW}\rangle$ i.e. $\langle \Omega_N | \Psi_a^{LW} \rangle$.
This overlap can be made to match exactly the partition function of well known families of integrable models!

But why does it work?????

categorical symmetry …..
holographic relation between CFT and TQFT ……

Ising model as an example

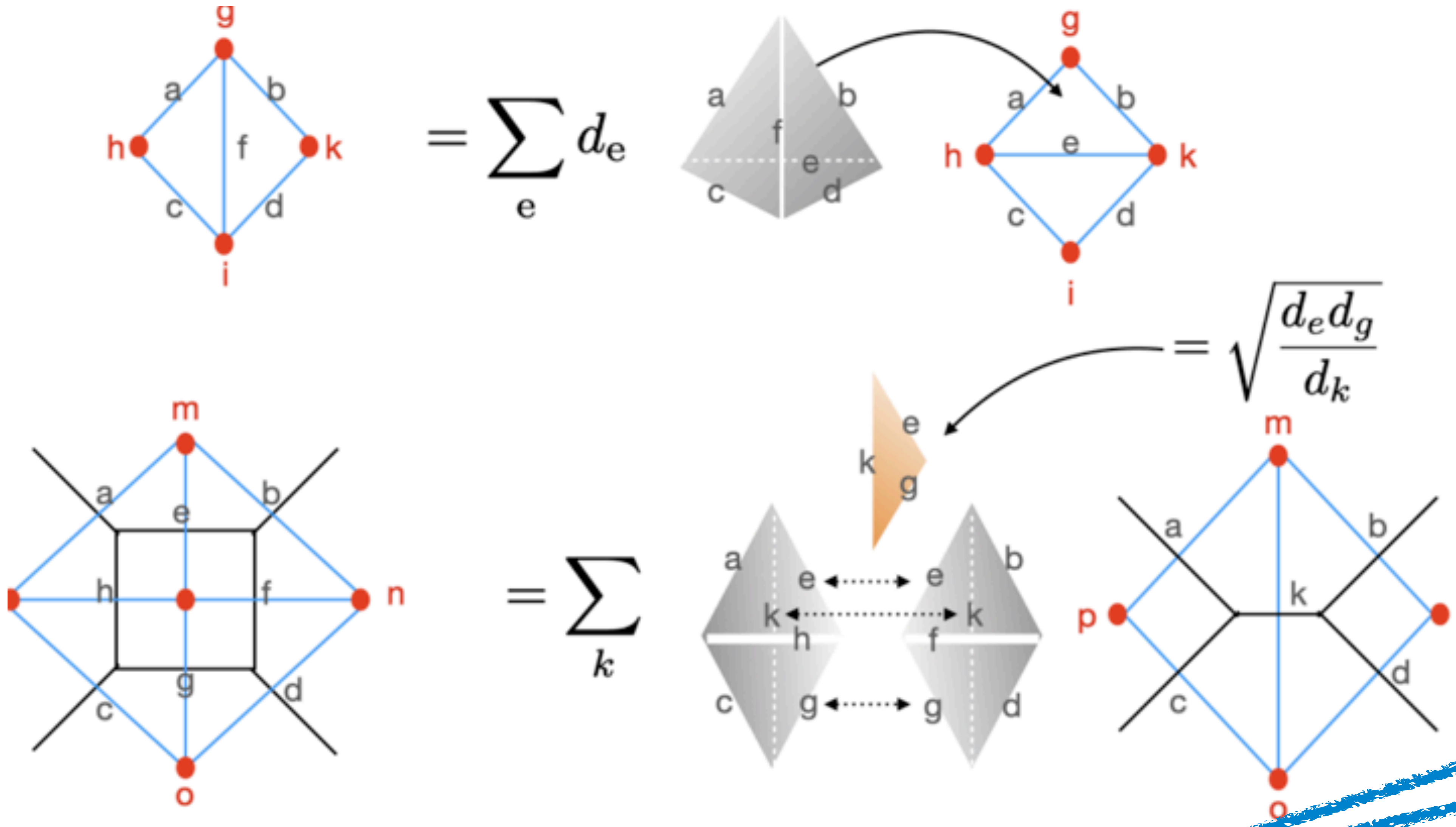


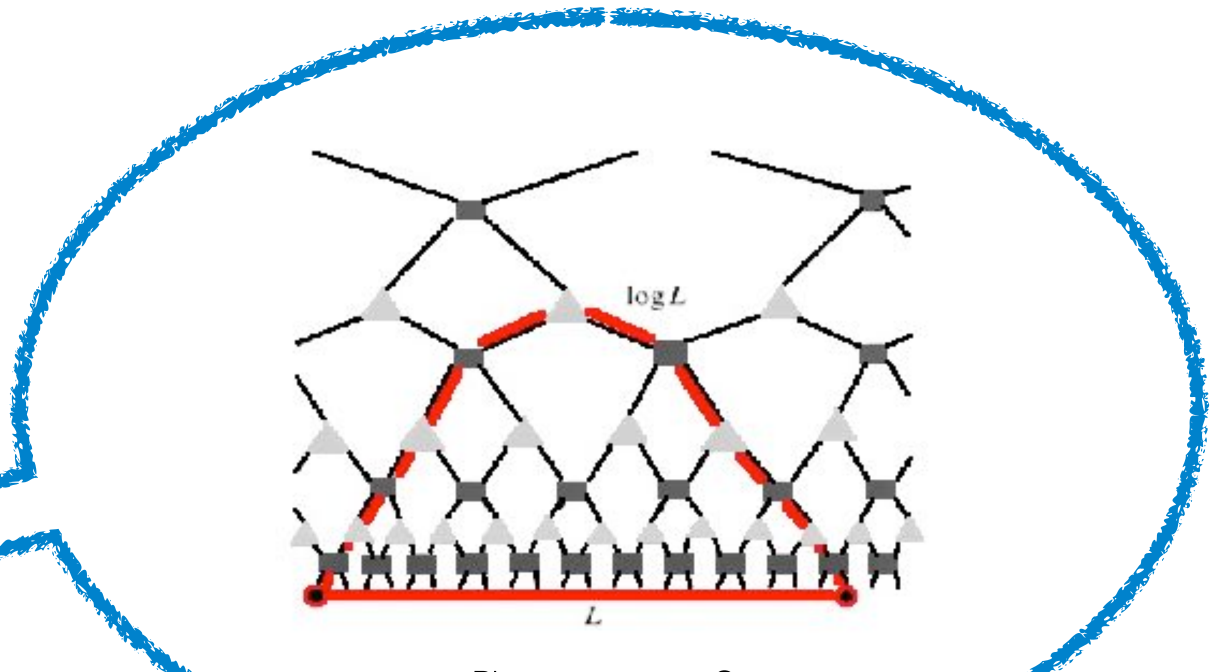
RG operator, holographic tensor networks and their fixed points

What boundary state $\langle\Omega_N|$ produces a CFT?

- For a given lattice on which the ground state wave-function is defined, it can be related to another wave-function on a different lattice using the pentagon equation and orthogonality condition of the 6j symbols:

Bal, Williamson, Vanhove, Bultinck, Haegeman, Verstraete 2017; Chen, Zhang, Wang, Ji, Zeng, Shen, LYH 2022





Picture courtesy Orus

$$\langle\Omega_N|\Psi_a^{LW}\rangle$$

$$\langle\Omega_N|FF|\Psi_{ka}^{LW}\rangle = \langle\Omega_{N-1}|\Psi_{ka}^{LW}\rangle$$

Here, we make the observation that recursively repeating this coarse-graining produces a collection of F's that geometrically fills up a Euclidean AdS3 and looks like a MERA. It is in fact an analytic holographic tensor network !

Finding Fixed Points of the RG operator

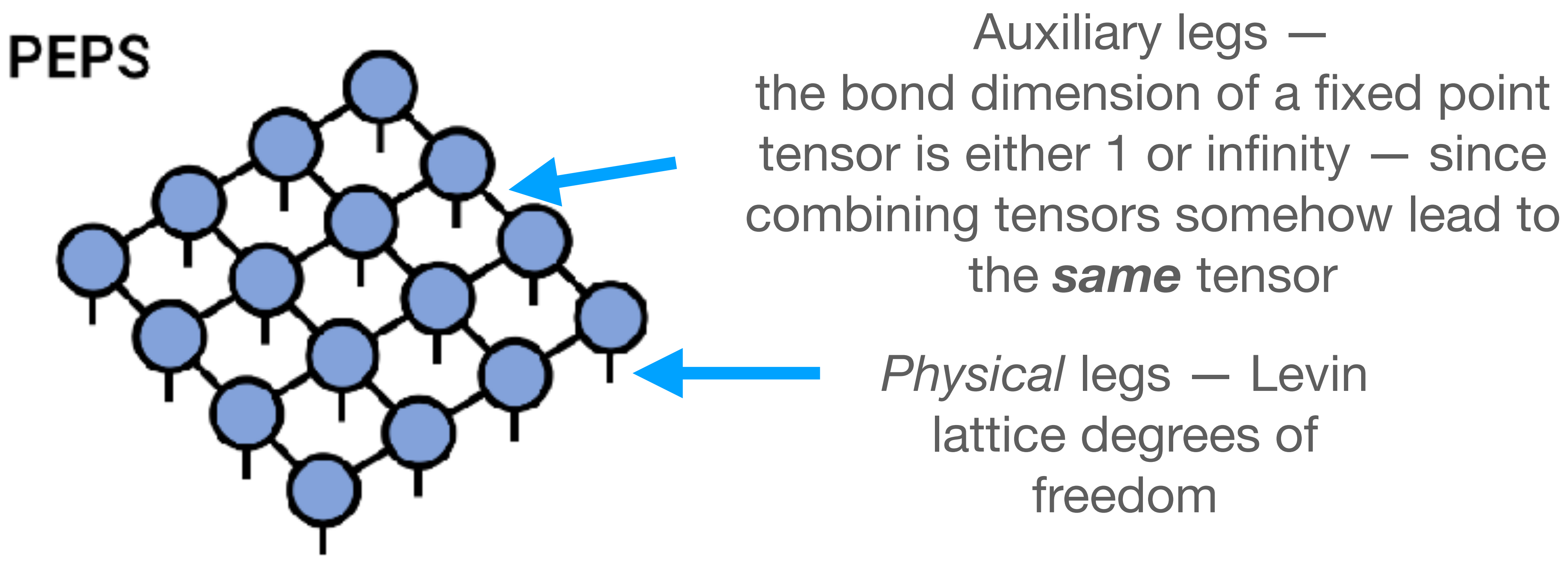
Constructing $\langle\Omega_N|$ as a PEP state

Chen, Zhang, Wang, Ji, Zeng, Shen, LYH 2022

- The direct product states can be understood as a seed state that could flow to the fixed points of the RG operator so that it recovers some scale invariant theory (including CFT and TQFT and more) in 2d

- Contracting the boundary conditions with the RG operator can be formulated as a TRG process described previously (except the process now explicitly preserves topological symmetry) — therefore finding eigenstates of the RG operator is equivalent to finding fixed point tensors of this topologically symmetric TRG

- If $\langle\Omega_N|$ is at the fixed point, we expect it to have entanglement rather than being a direct product state. But since we are interested in local CFT's, the entanglement of $\langle\Omega_N|$ should be local. We should construct it using a PEP tensor network
i.e.

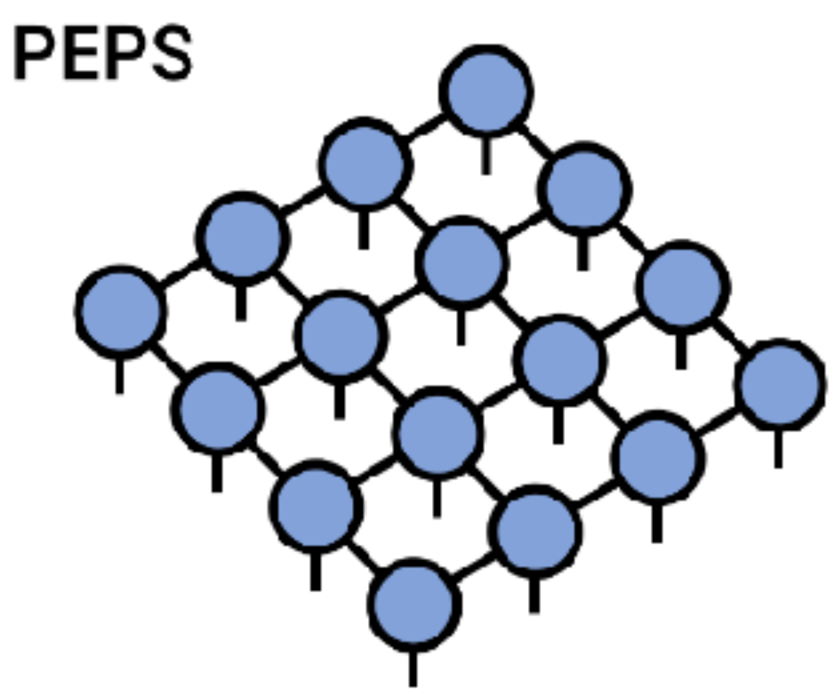


Picture courtesy https://tensornetwork.org/

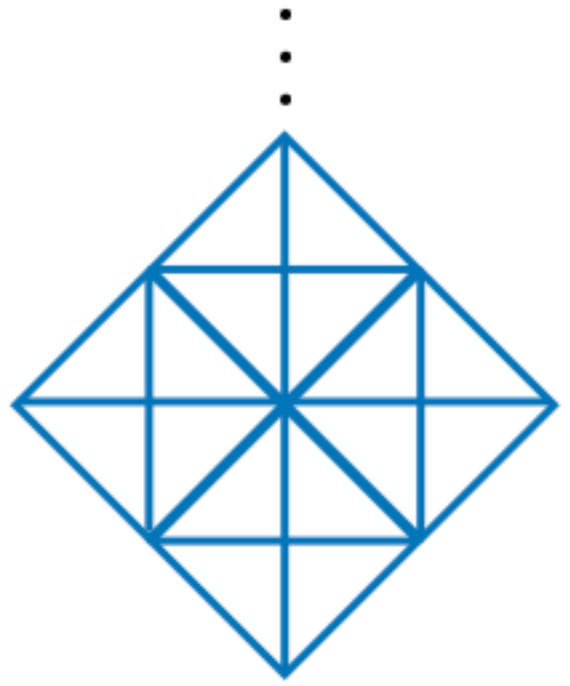
Finding Fixed Points of the RG operator

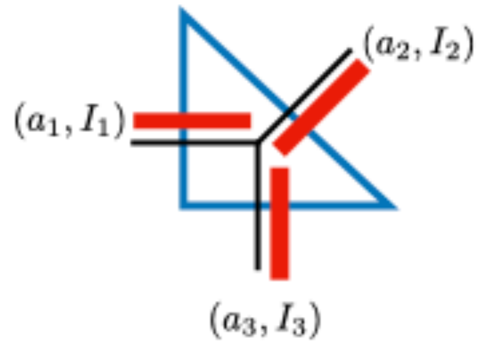
Constructing $\langle\Omega_N|$ as a PEP state - solutions corresponding to 2D TQFT

Chen, Zhang, Wang, Ji, Zeng, Shen, LYH 2022



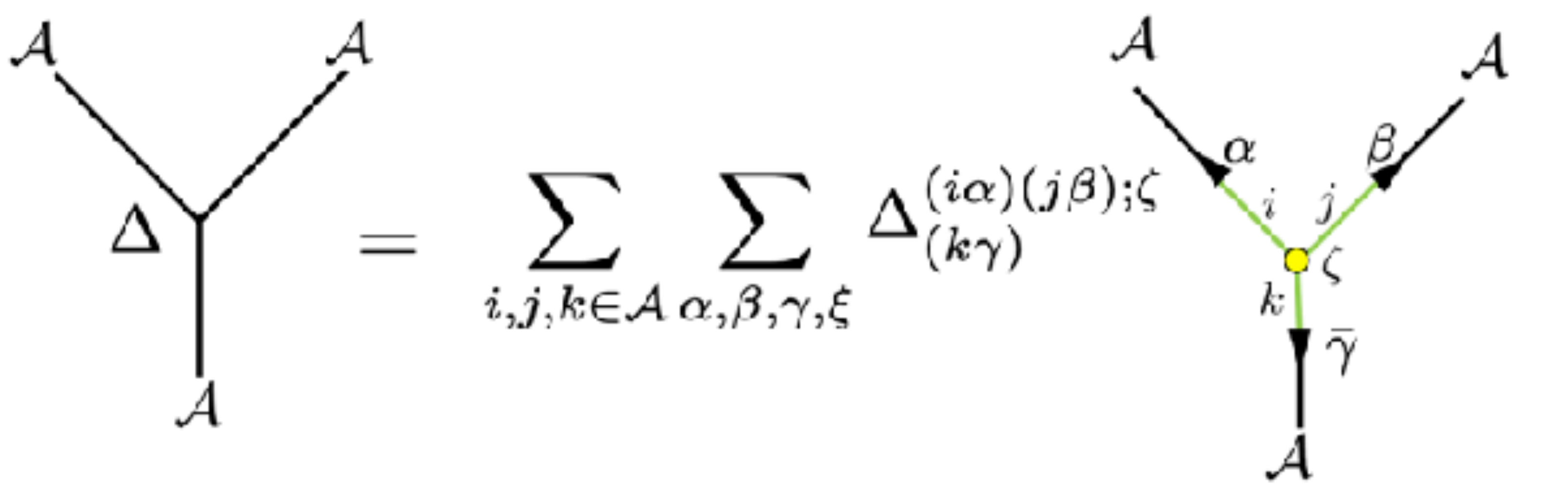
Our ansatz:

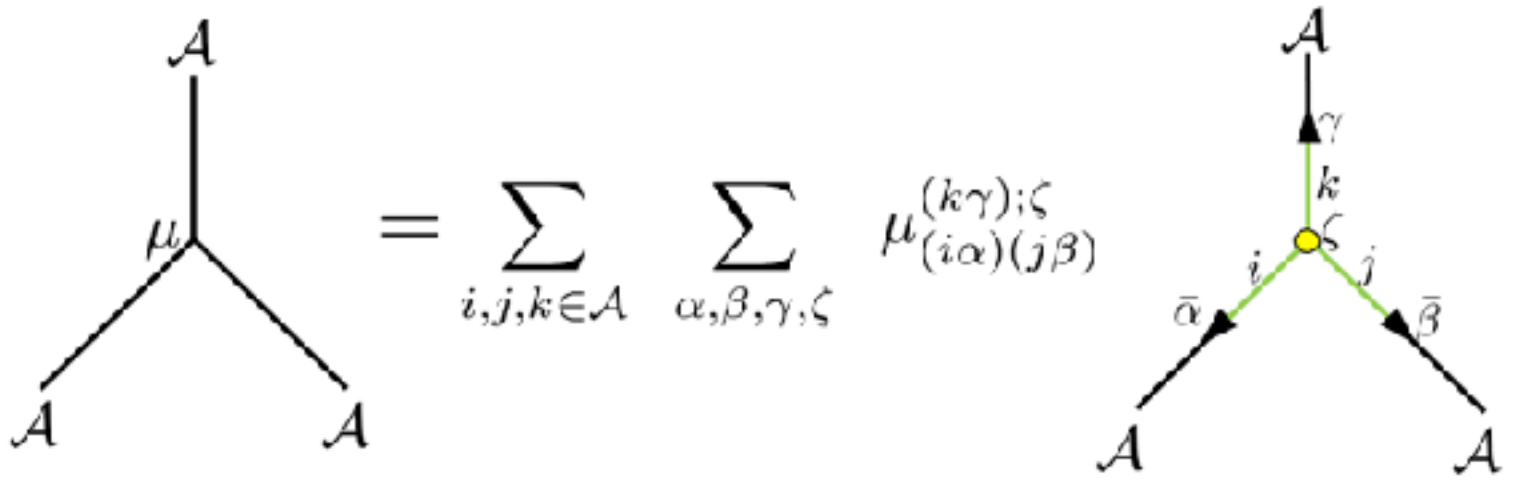
$\langle\Omega| = \cdots$  $\cdots$

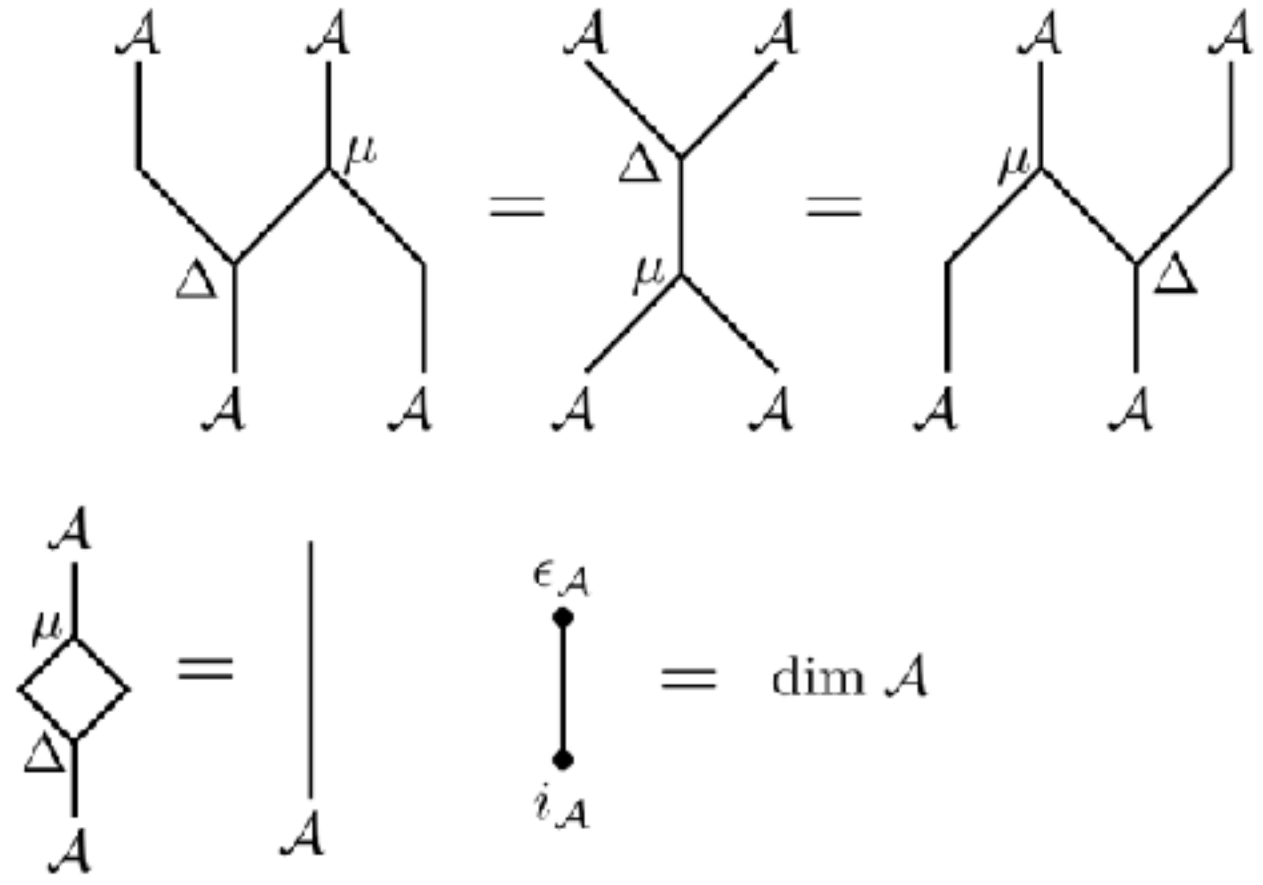
$T^{a_1a_2a_3}_{I_1I_2I_3} =$ 

$$T^{a_1a_2a_3}_{I_1I_2I_3} = \mu^{(a_3I_3)}_{(a_1I_1)(a_2I_2)}$$ gives topological fixed point

Frobenius algebra







Can recover the partition functions of the 2d Symmetry Protected Topological states (SPT) if the RG operator came from 3d Dijkgraaf Witten Theory

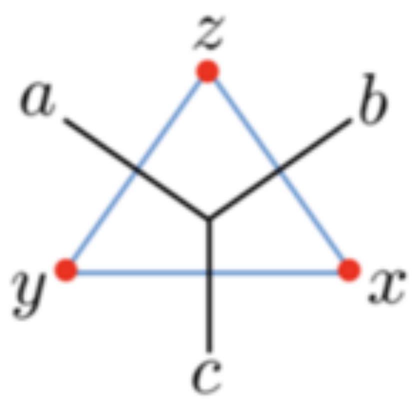
Finding Fixed Points of the RG operator

Where are the CFT's ?

- They can be viewed as phase transition points between the topological TQFT
  Start with an interpolation of topological boundary fixed points, and use the RG operator on it repeatedly. When far from the phase transition points they flow to either one of the boundary fixed points. At phase transitions the RG operator gets confused — could give a CFT fixed points

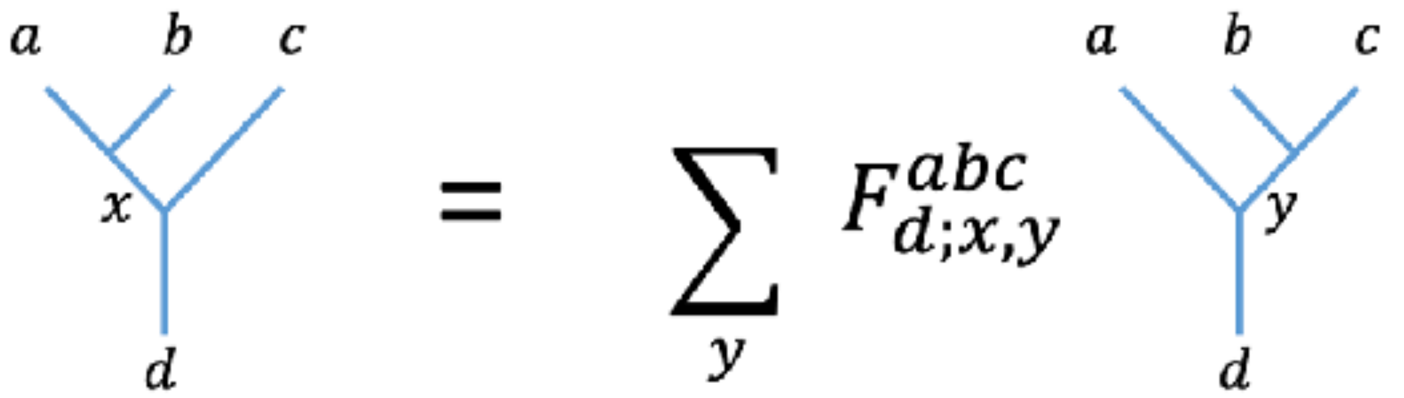
k	A1/A0— theoretical	Our numerics
2	0.643594	0.60-0.61
3	0.697043	0.67-0.68
4	0.719471	0.69-0.70
5	0.731426	0.71-0.72
6	0.738656	0.72-0.73

Fixed Point Boundary Corresponding to CFT?

- Note that 

$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F_y^{abx})^*_{cz}$$

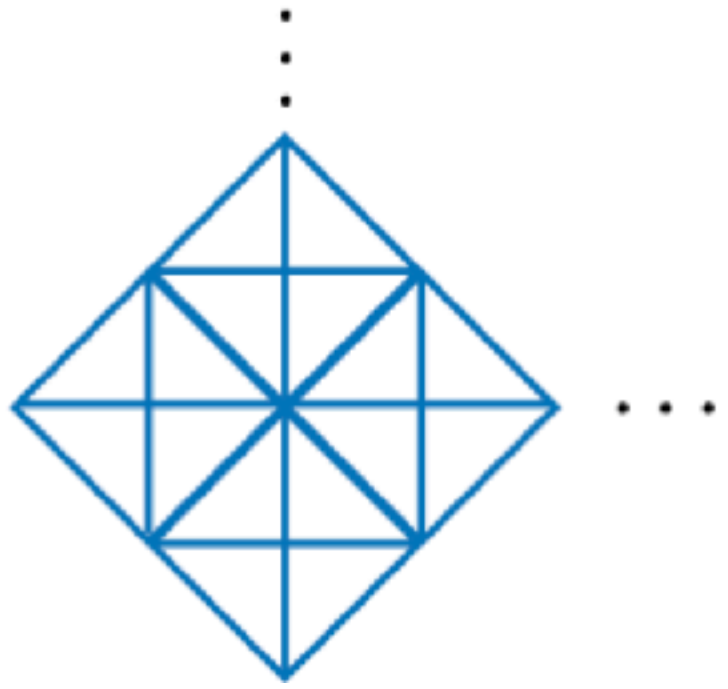
this is proportional to the open string (boundary operator) fusion coefficient in a diagonal rational eft characterised by the tensor category! see Fuchs, Runkel Schweigert 2002 and series of papers

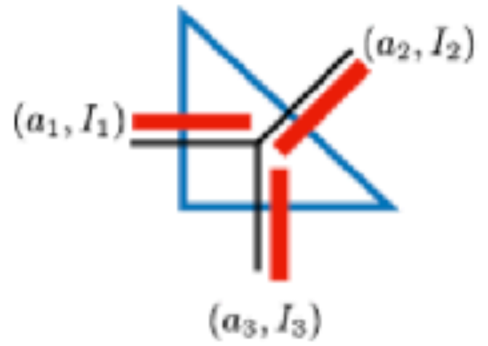
Also, it is well-known that conformal blocks transform as 

(where a,b,c,d,x,y are the labels of the families of primary representations of the)

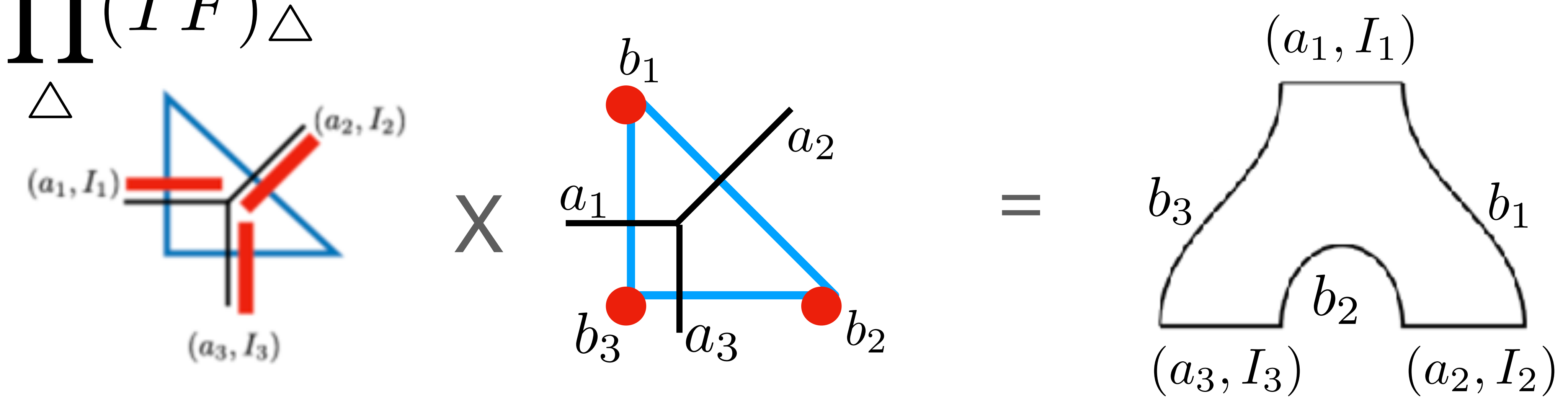
Therefore ~~~

Fixed Point Boundary Corresponding to CFT?

$\langle \Omega | = \cdots$  $\cdots$

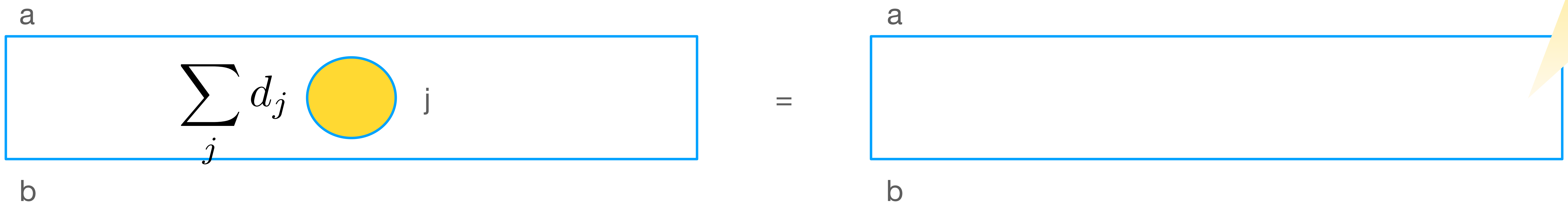
$T^{a_1 a_2 a_3}_{I_1 I_2 I_3} =$ 

$$\langle \Omega | \Psi \rangle = \prod_{\triangle} (TF)_{\triangle}$$



is a fixed point of the RG operator if T is basically the three point function between chiral operators belonging to the family a1,a2,a3. The I1, I2, I3 auxiliary indices would then correspond labels of the descendent of the given chiral family subjected to the condition :

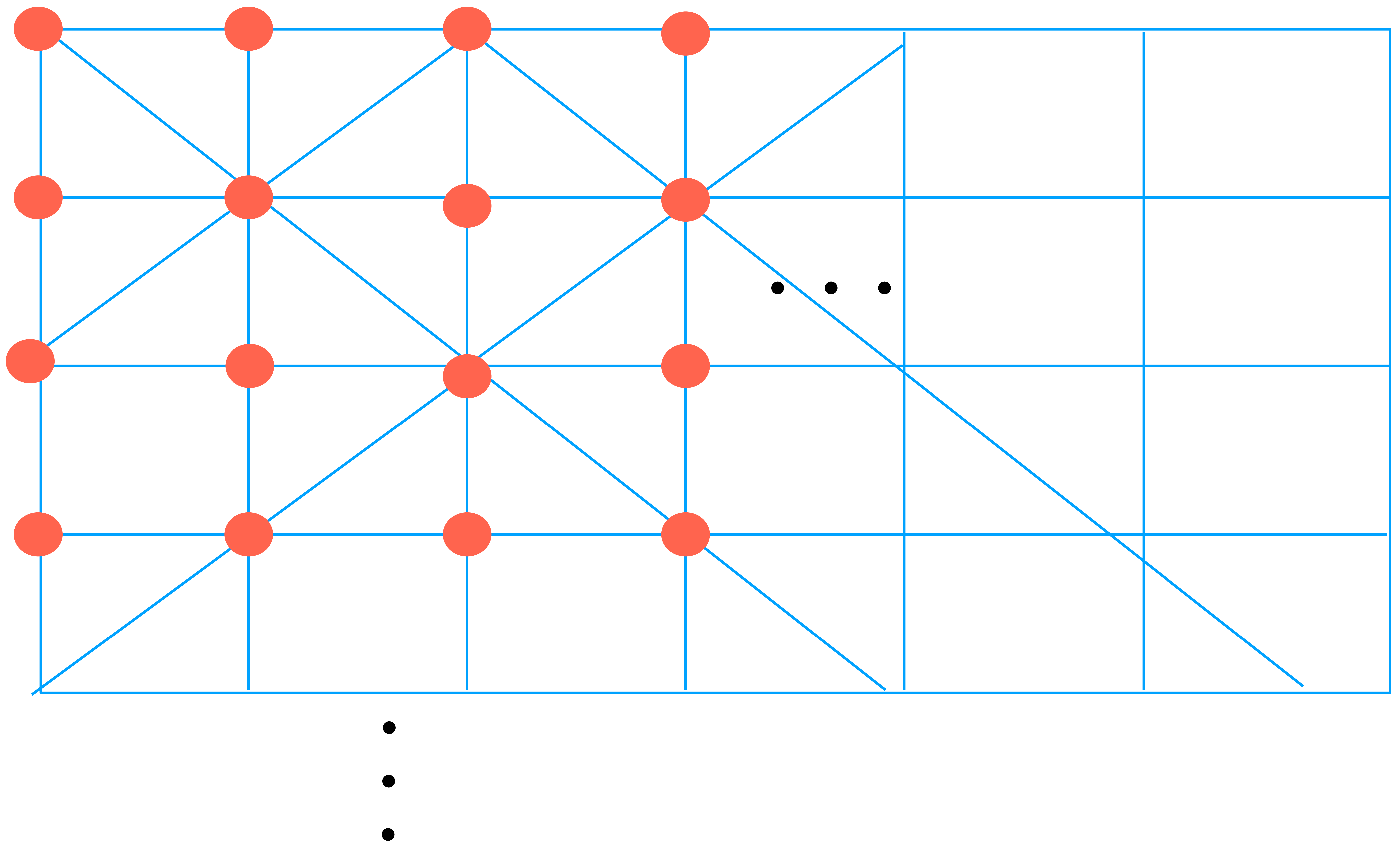
We checked this for Ising model numerically and find up to 4 sigfig accuracy with a truncated bond dimensions



The Turaev -Viro TQFT would dictate that the conformal boundary condition is ``closeable” in exactly the same way as the proposed “entanglement brane boundary condition” . Topological defects can ***pass through this boundary as if they are transparent.***

LYH, Wong 2020; Brehm, Ruunkel 2022

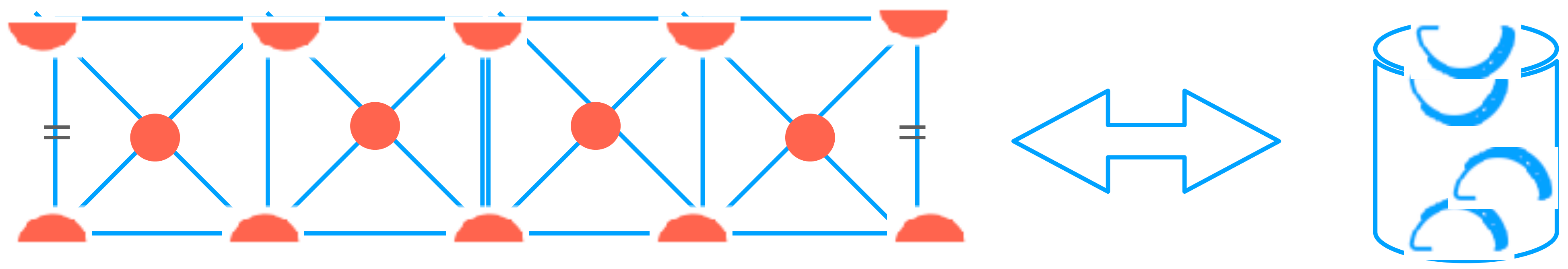
Factorising CFT partition function



The Turaev-Viro TQFT construction can describe continuous field theory because the fixed point tensors can be sewed together and finally the holes can be contracted — the connection between lattice construction and continuous theory can be understood in this light. This picture can be generalised to non-diagonal RCFT.

Cylinder

recovering the closed spectrum of the CFT



- These half circles at the boundary allow them to be part of the index (ie don't sum at the boundary.

- Take Ising as an example again. We recover (keeping only 4-5 descendants at each primary sector) the following spectrum — seems pretty decent!

s	$(\Delta_{TH}, \Delta_{Num})$			
0	(0.125, 0.127)	(1.0, 1.035)	(2.125, 2.259)	(3.0, 3.253)
1	(1.125, 1.129)	(2.0. 2.09)	(3.125, 3.184)	(4, 3.724)
2	(2.0, 2.0)	(2.0, 2.02)	(2.125, 2.211)	(2.125, 2.211)
3	(3.125, 3.230)	(3.125, 3.330)		

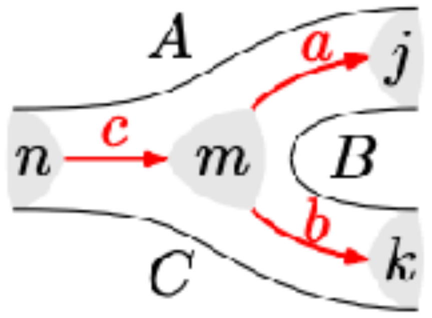
Remark: Non-diagonal Theory

Alternative tensor network representation of the Levin-Wen ground state

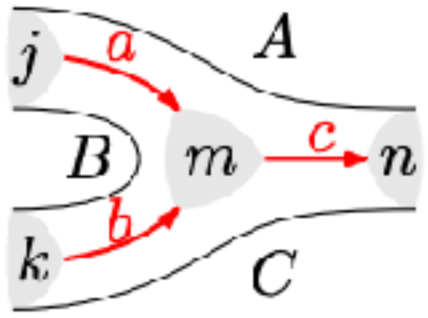
Lootens, Fuchs, Haegeman, Schweigert, Verstraete, 2021

2.1 Bimodule categories

At this point, we have illustrated that MPO symmetries of PEPS define two fusion structures (more correct terminology: monoidal structures) corresponding to horizontal fusion of MPO products, and vertical fusion related to MPO scale transformations. They thus encode the algebraic data of two fusion categories $\mathcal{C}$ and $\mathcal{D}$. A particular prescription for such MPO tensors can be constructed by also invoking the algebraic data associated with a $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$ (see appendix B)[2]. In particular, the pulling-through condition, the zipper condition, the two recoupling identities and the two pentagon equations coincide with the different pentagon equations of $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{M}$ if we make the following identifications:



$$:= \left(\frac{d_a d_b}{d_c}\right)^{\frac{1}{4}} \frac{\left({}^{1}F^{abC}_{A}\right)^{B,kj}_{c,mn}}{\sqrt{d_B}},$$



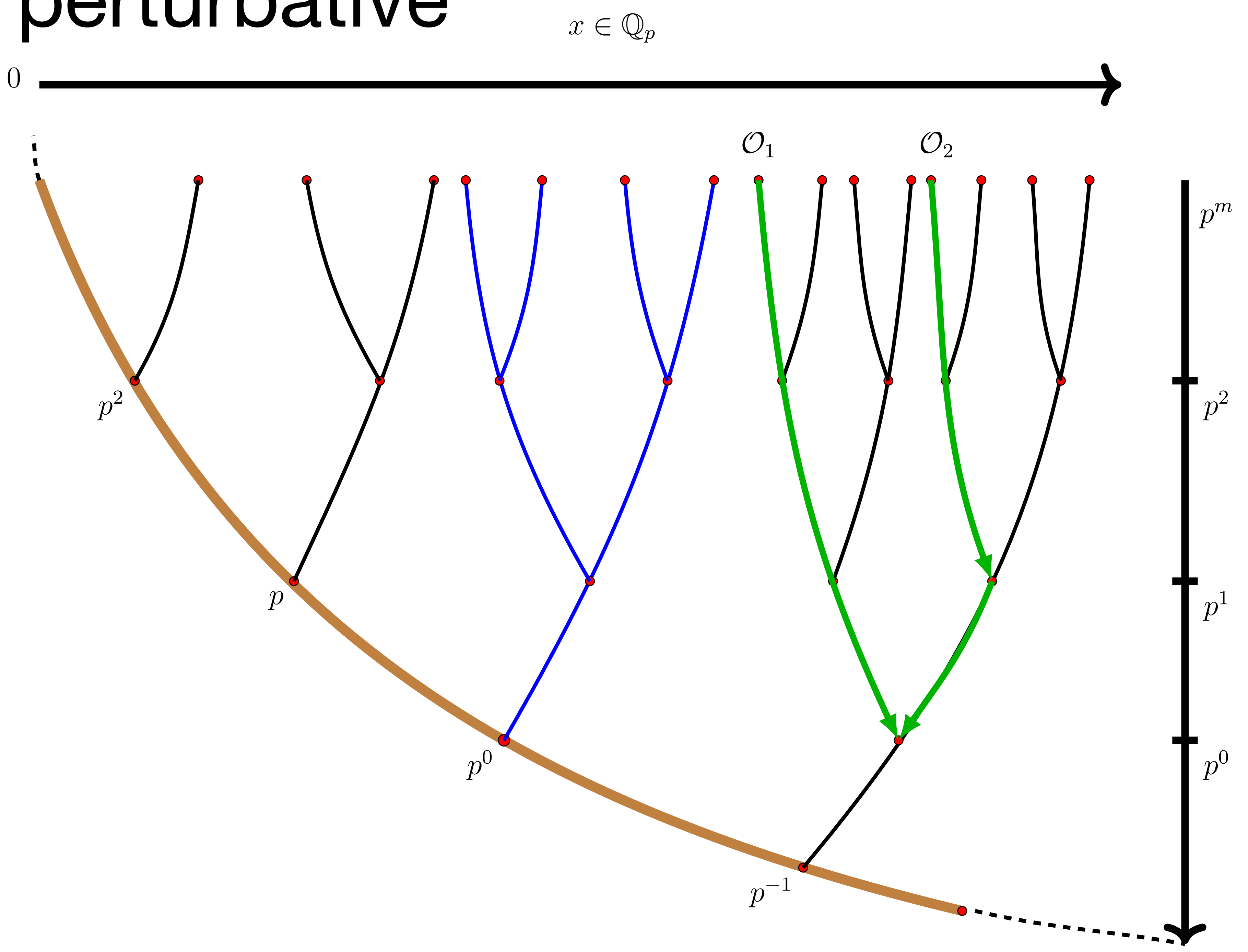
$$:= \left(\frac{d_a d_b}{d_c}\right)^{\frac{1}{4}} \frac{\left({}_{1}F^{abC}_{A}\right)^{B,kj}_{c,mn}}{\sqrt{d_B}},$$

This gives explicit prediction of contractible boundary conditions for non-diagonal theories. The boundaries are labeled by objects in the module category, and the weights is the quantum dimension of these objects.

Relation with p-adic tensor network

W. Li, C. Melby-Thompson, LYH 2019

- Tensor Network that recovers the p-adic CFT and reproduces (some) parts of the AdS/CFT dictionary and a perturbative



	p-adic Tensor Network	Turaev-Viro type tensor network
tensor components	structure coefficients of CFT	open structure coefficients of CFT
algebraic structure of tensors	Frobenius/associative algebra	(1)-fusion category
CFT dimensions	(quasi) 1-dimension CFT	2 dimensional CFT
Bulk dimensions	(quasi) 2 dimensions	3 dimensions

Liouville theory

Reconstruction from the open structure coefficients

Open Liouville Theory

Putting together the closed partition function from the open one

- Open structure coefficients of Liouville theory has been considered.

$$\tilde{C}^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_3,\beta_2,\beta_1} = \frac{1}{\sqrt{\gamma_0}\,\Gamma_b(Q)}\sqrt{\frac{|S_b(2\beta_3)|^2}{|S_b(2\sigma_2)|^4}}\sqrt{\tilde{C}(\beta_3,\beta_2,\beta_1)}\begin{Bmatrix}\sigma_1 & \beta_1 & \sigma_2\\ \beta_2 & Q-\sigma_3 & \beta_3\end{Bmatrix}_b$$

← 6j symbols of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

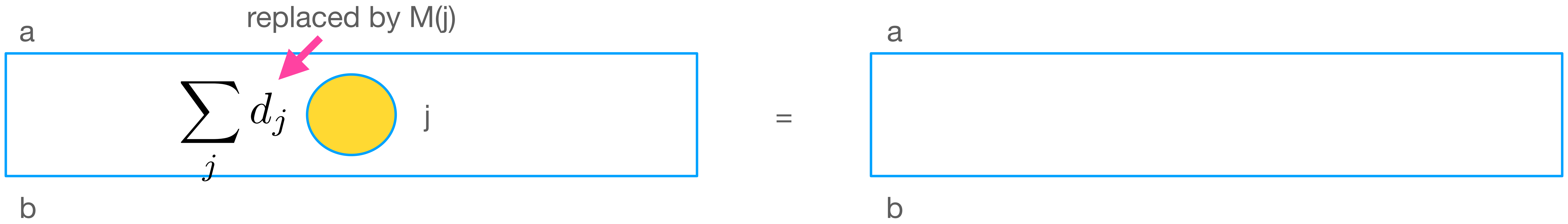
Pentagon + orthonality condition can be used to generate an RG operator that fills an AdS like space. This is a kind of Turaev Viro formulation "closely related" to the Kashaev form of Teichmuller TQFT which is in turn related to AdS3 gravity

$$\int_{Q/2+i\mathbb{R}^+} d\mu(\delta_1)\begin{Bmatrix}\alpha_1 & \alpha_2 & \beta_1\\ \alpha_3 & \beta_2 & \delta_1\end{Bmatrix}_b\begin{Bmatrix}\alpha_1 & \delta_1 & \beta_2\\ \alpha_4 & \alpha_5 & \gamma_2\end{Bmatrix}_b\begin{Bmatrix}\alpha_2 & \alpha_3 & \delta_1\\ \alpha_4 & \gamma_2 & \gamma_1\end{Bmatrix}_b = \begin{Bmatrix}\beta_1 & \alpha_3 & \beta_2\\ \alpha_4 & \alpha_5 & \gamma_1\end{Bmatrix}_b\begin{Bmatrix}\alpha_1 & \alpha_2 & \beta_1\\ \gamma_1 & \alpha_5 & \gamma_2\end{Bmatrix}_b$$

$$\int_{Q/2+i\mathbb{R}^+} d\mu(\alpha_s)\begin{Bmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{Bmatrix}^*_b\begin{Bmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha'_t\end{Bmatrix}_b = (M(\alpha_t))^{-1}\delta(\alpha_t-\alpha'_t)\,.$$

$$d\mu(\alpha) = d\alpha\, M(\alpha)\,, \qquad M(\alpha) := |S_b(2\alpha)|^2\,.$$

This orthogonality suggests that the Liouville shrinkable boundary condition is given by:



Open Liouville Theory

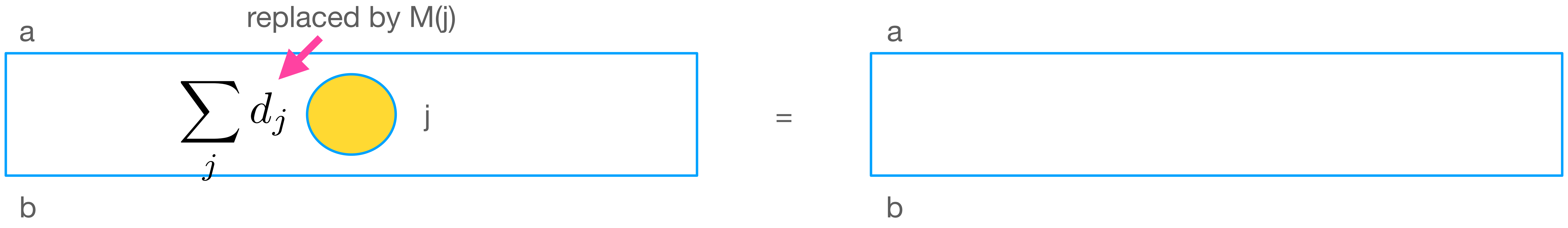
Putting together the closed partition function from the open one

- Some recent discussions of the TQFT dual of the Liouville theory

Collier, Eberhardt, Zhang 2023; Belin, de Boer, Nayak, Sonner 2023 (mentioning the simplicial formulation that is different from the current discussion); Ubaldo, Perlmutter 2023

$$d\mu(\alpha) = d\alpha \, M(\alpha)\,, \qquad M(\alpha) := |S_b(2\alpha)|^2\,.$$

This orthogonality suggests that the Liouville shrinkable boundary condition is given by:



Outlook

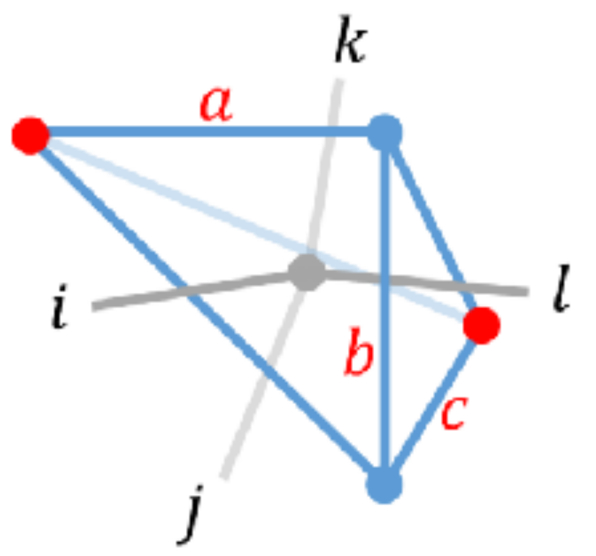
- Large C CFTs
- Supersymmetric generalisations — include fermions
- Can we read off something about black holes, or anything useful from this holographic tensor network ?
- Higher d CFTs — Ising CFT for example can definitely be constructed in this way, we should construct some more examples

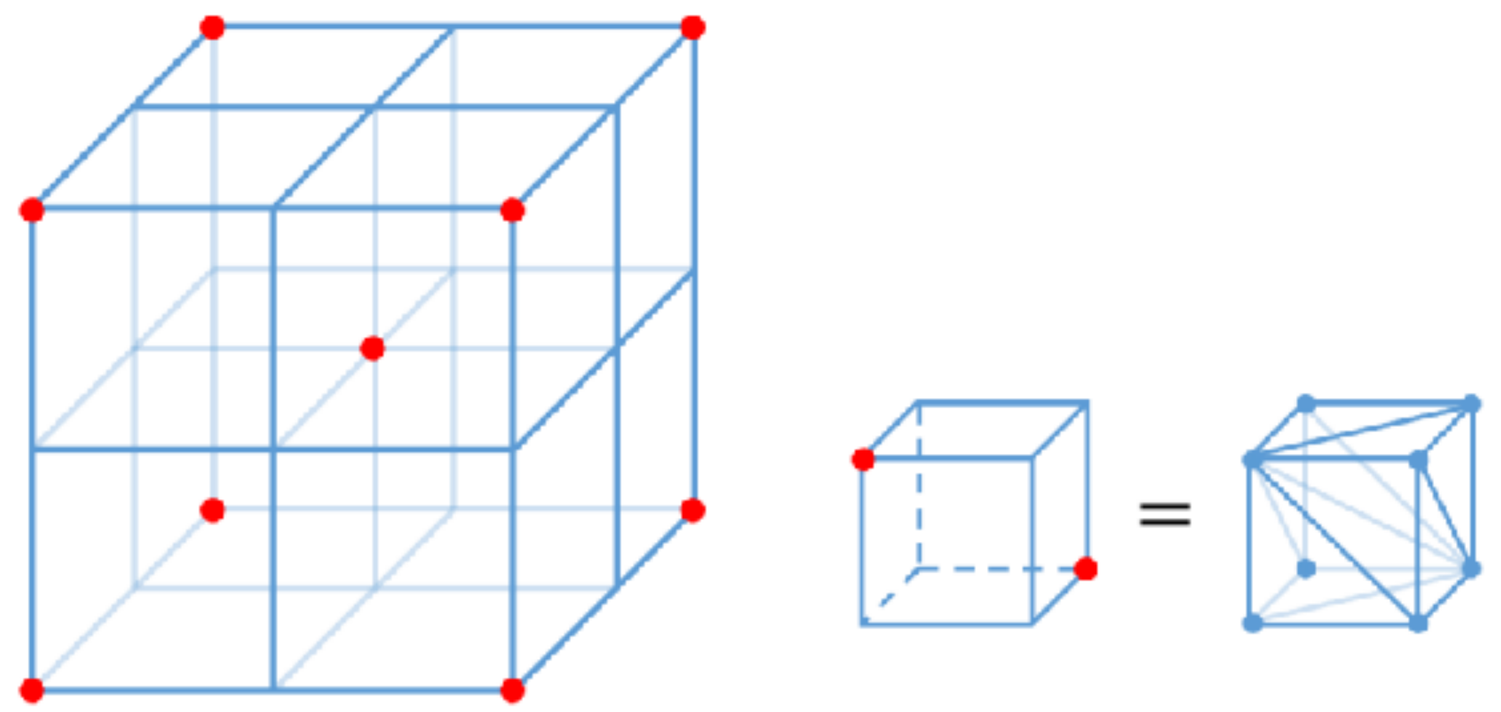
  Chen, Zhang, Wang, Ji, Zeng, Shen, LYH 2022
- Perhaps not necessarily related to (topological)AdS/CFT? or perhaps it still bear some relation as in the case of 3d (i.e. boundary condition does break the topological invariance to some extent.., and we need to work with irrational TQFT…)

3+1 D TQFT and 2+1 D CFT

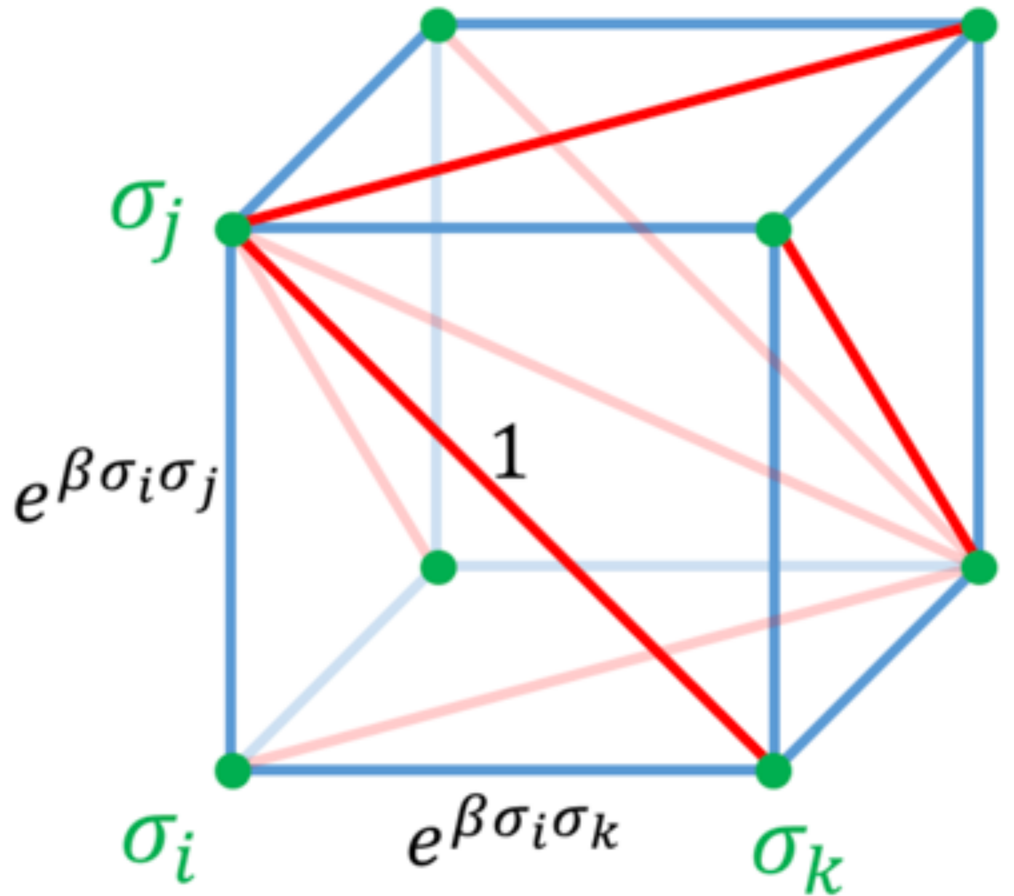
Example: 4D Z2 Dijkgraaf - Witten Theory and the Ising model

- Tensor Network Representation of the ground state wave-function of Dijkgraaf-Witten theory:





Boundary conditions for the Ising model:



3+1D DW Theories and RG operator

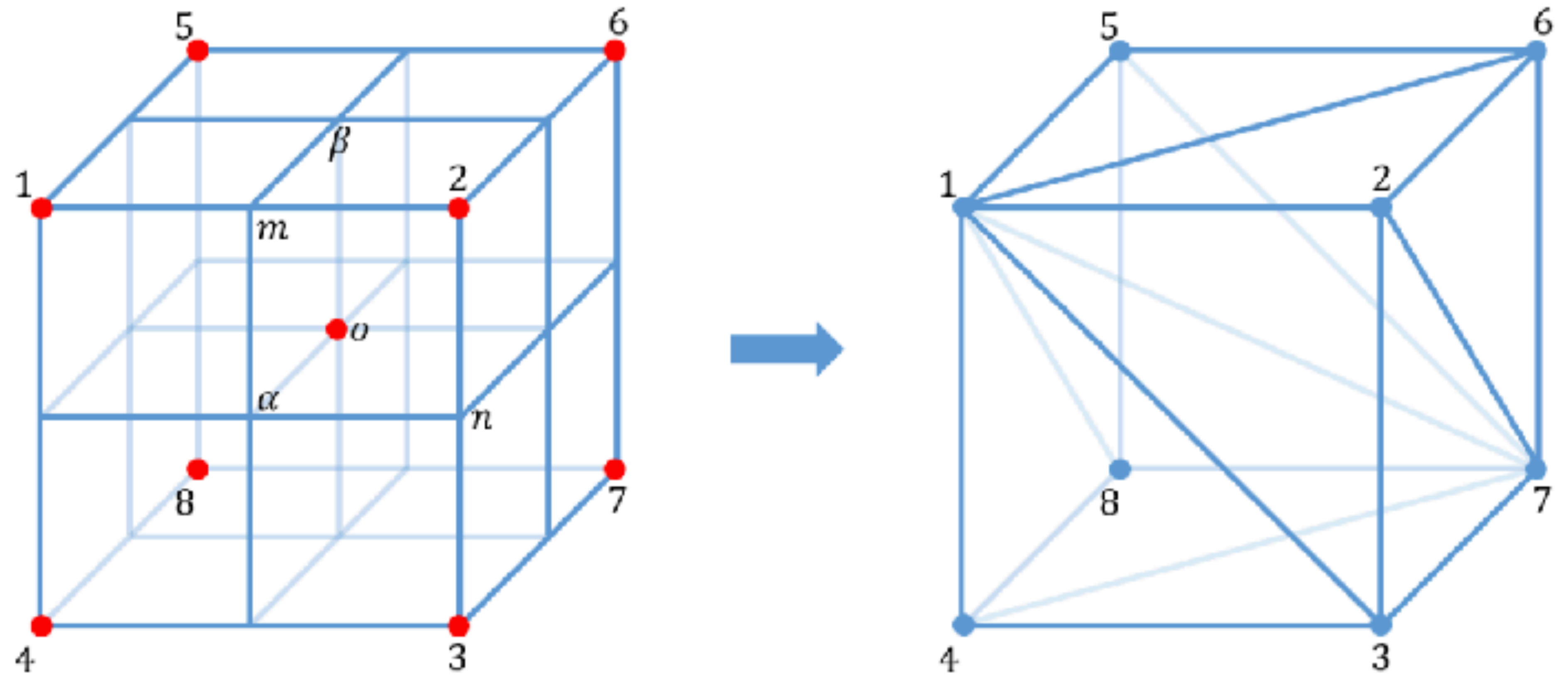


FIG. 17. Coarse grain the $2 \times 2 \times 2$ cube into $1 \times 1 \times 1$ cube.

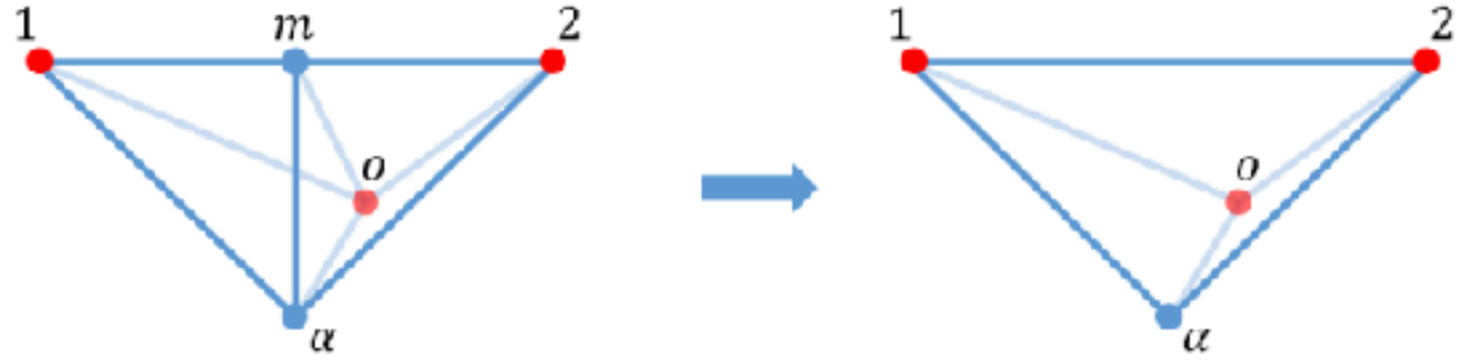


FIG. 18. Combine $1m\alpha o$ and $2m\alpha o$ to get a bigger tetrahedron $12\alpha o$.

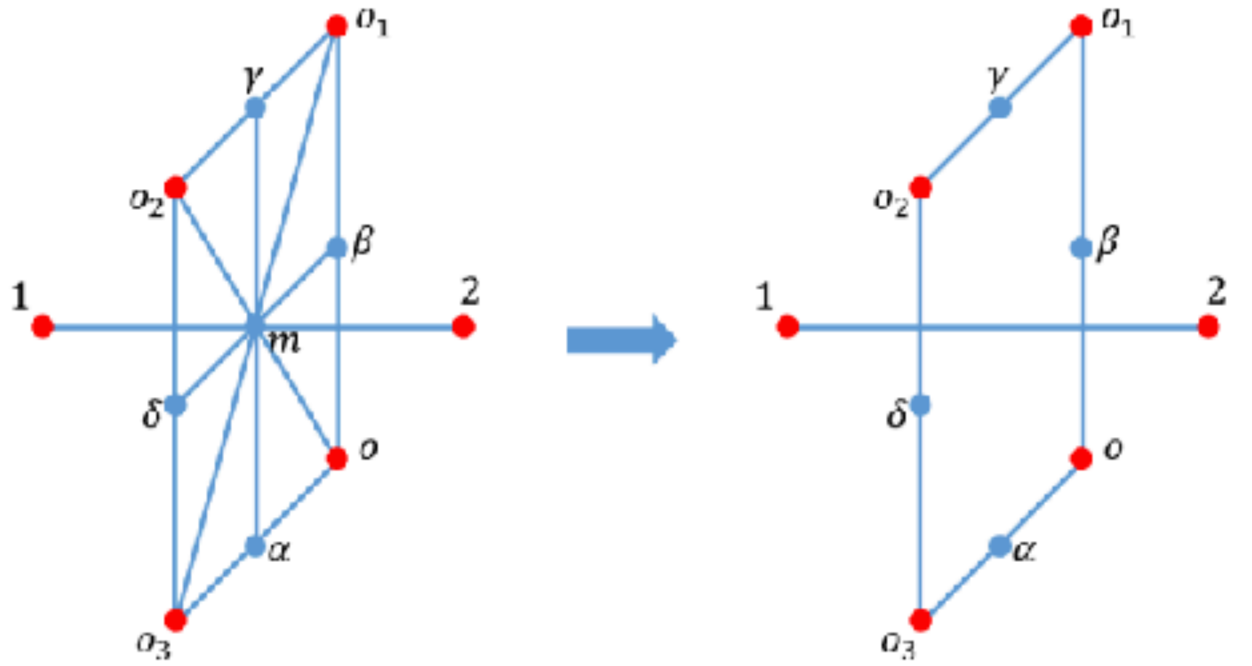


FIG. 19. The first step is to eliminate the vertices like m and to obtain edges like 12. To avoid clutter, we omit the edges connecting $1, 2$ with $\alpha, \beta, \gamma, \delta, o, o_1, o_2, o_3$. On the left hand side, there is a vertex m, and there are 16 small tetrahedron $1m\alpha o, 1m\beta o, 1m\beta o_1, \ldots, 2m\alpha o, 2m\beta o, 2m\beta o_1 \ldots$. On the right hand side, there is no m, and there are 8 bigger tetrahedron $12\alpha o, 12\beta o, 12\beta o_1, \ldots$. They are on the two boundaries of a 4D body which consists of eight 4-simplices $12m\alpha o$, $12m\beta o$, $12m\beta o_1$, $12m\gamma o_1$, $12m\gamma o_2$, $12m\delta o_2$, $12m\delta o_3$, $12m\alpha o_3$.

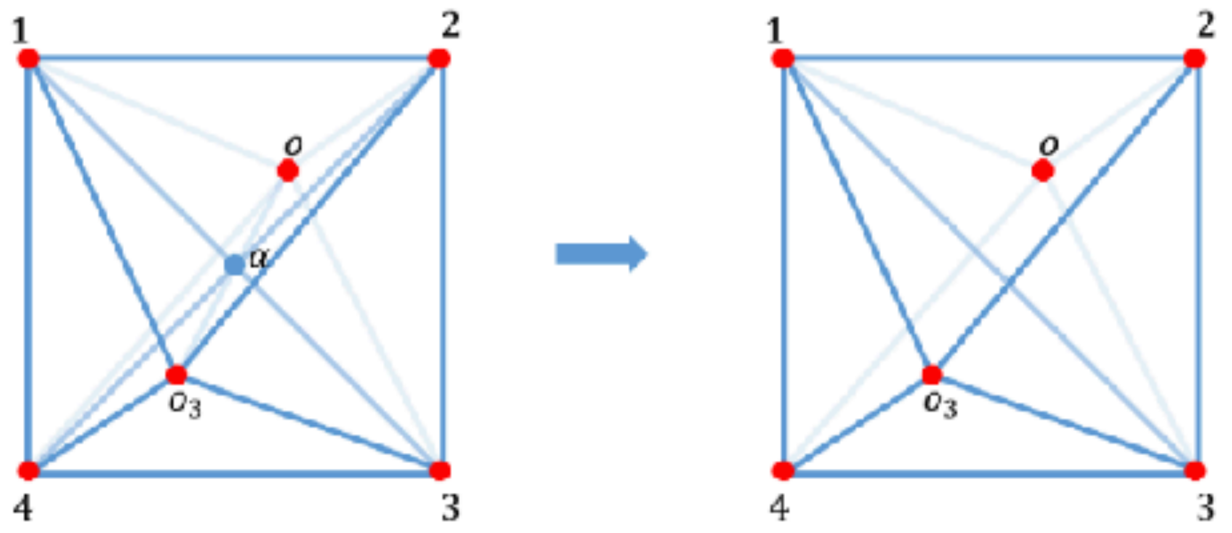


FIG. 21. The second step is to eliminate the vertices like α. Here we choose to connect vertices $1,3$ since in the target coarse grained cubic there is a 13 edge as shown in figure 17. On the left hand side, there is a vertex α, and there are 8 tetrahedra $12\alpha o, 23\alpha o, 34\alpha o, 41\alpha o, 12\alpha o_3, 23\alpha o_3, 34\alpha o_3, 41\alpha o_3$. On the right hand side, there is no α, and there are 4 tetrahedra $123o, 341o, 123o_3, 341o_3$. They are on the two boundaries of a 4D body which consists of four 4-simplices $123\alpha o, 123\alpha o_3, 341\alpha o_3, 341\alpha o$.

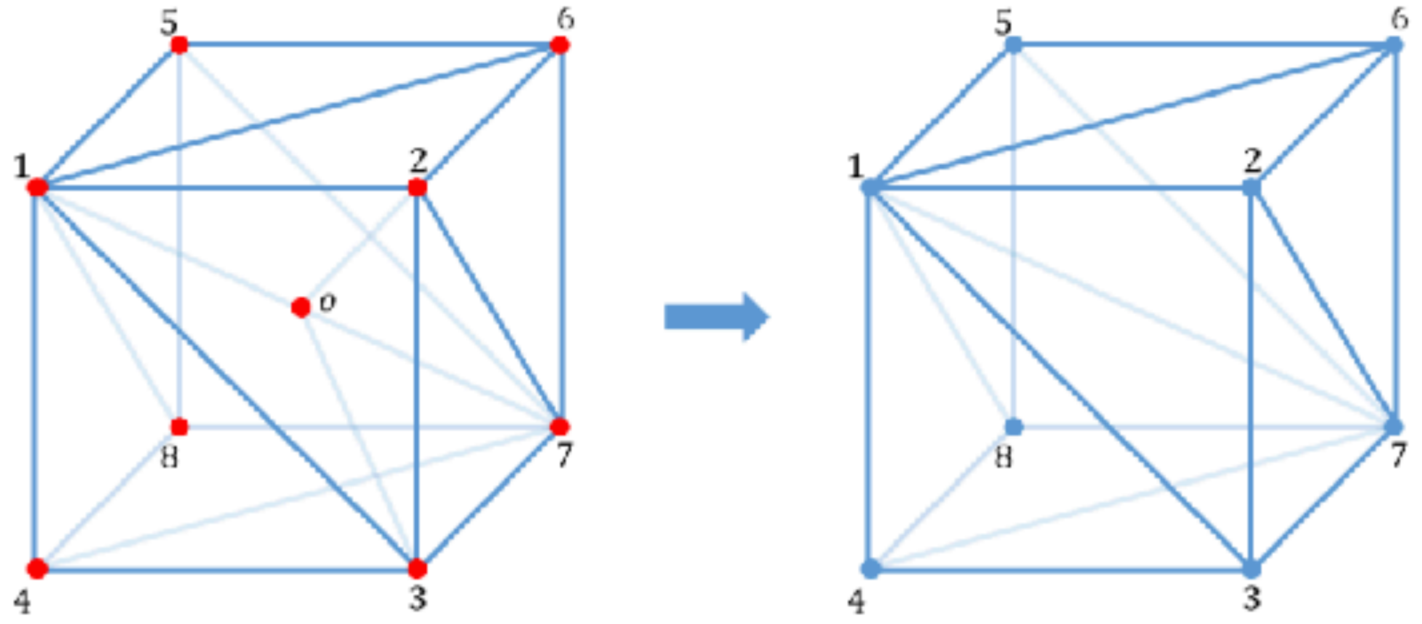


FIG. 22. The third step is to eliminate the vertex o and to obtain the edge 17. To avoid clutter, we only show some of the edges connecting o with $1,2,3,4,5,6,7,8$. On the left hand side, there is a vertex o, and there are 12 tetrahedra $123o, 143o$, $237o, 267o$, $126o, 156o$, $148o, 158o$, $487o, 437o$, $567o, 587o$. On the right hand side, there is no o, and there are 6 tetrahedra 1237, 1267, 1567, 1587, 1487, 1437. They are on the two boundaries of a 4D body which consists of six 4-simplices $1237o, 1267o, 1567o, 1587o, 1487o, 1437o$. Combining $123o$ and $237o$ to get the tetrahedron 1237 can be read off from this figure.

Topological solutions = Higher Frobenius Algebra

Wang, Li, Hu, Wan, *JHEP* 10 (2018) 114,   Zhao, Lou, Zhang, Hung, Kong, Tian, [2208.07865]

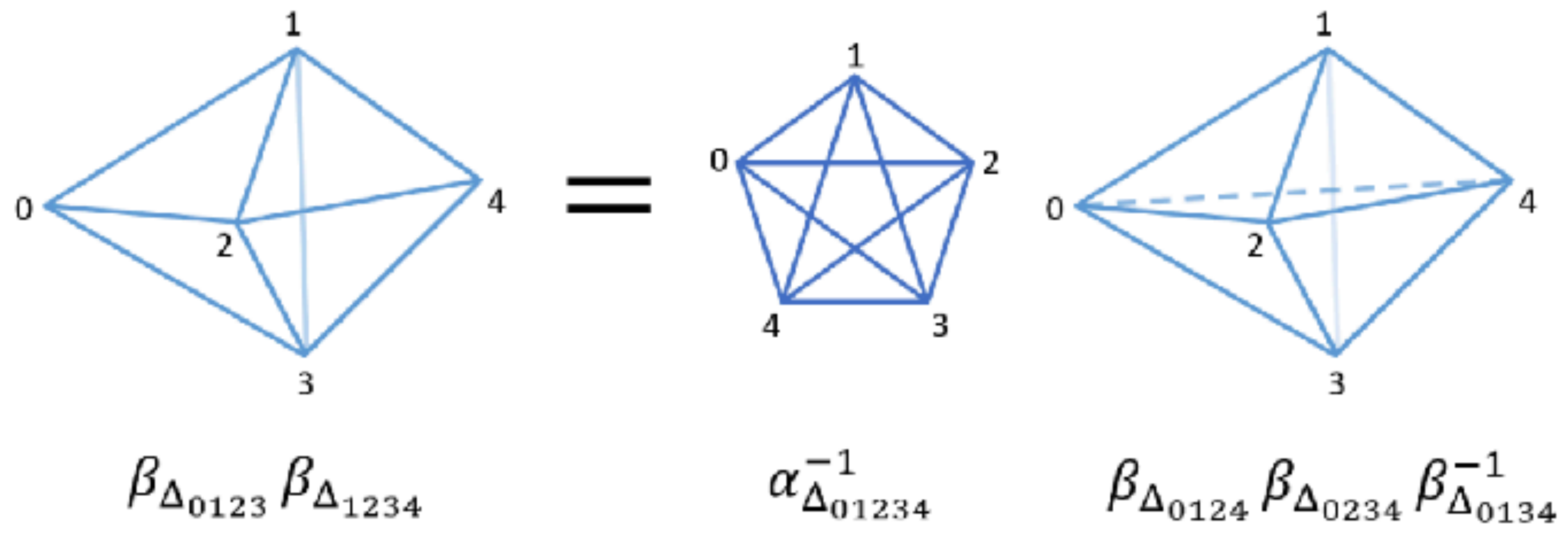


FIG. 23. There are 2 tetrahedra on the left and 3 tetrahedra on the right corresponding to two different triangulations of the boundary. We have $\beta_{\Delta_{0123}}\beta_{\Delta_{1234}} = \alpha^{-1}_{\Delta_{01234}}\beta_{\Delta_{0124}}\beta_{\Delta_{0234}}\beta^{-1}_{\Delta_{0134}}$. The powers of -1 are related to the orientations.

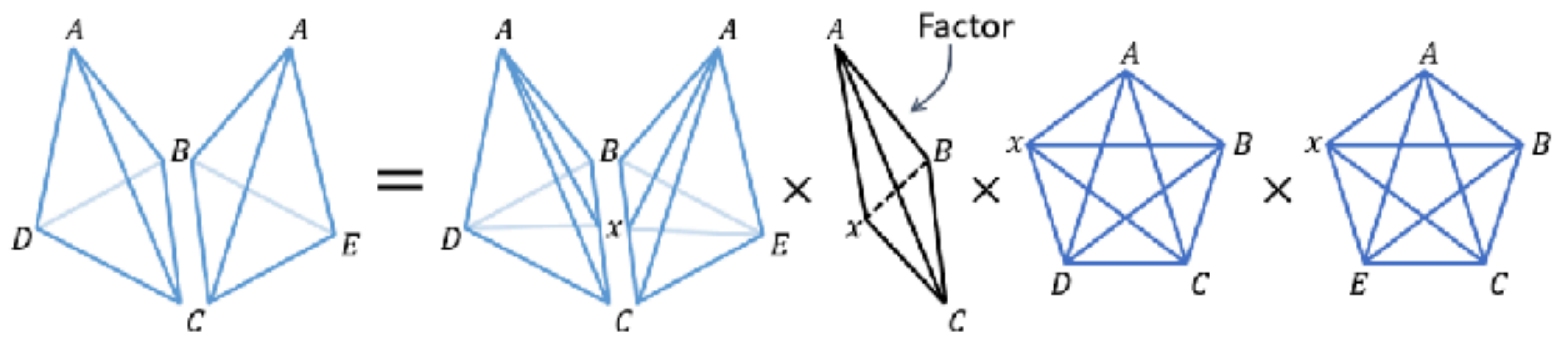


FIG. 24. The blue tetraheron corresponds to the boundary factors β. The pair of 4-simplices on the right hand side corresponds to the 4-cocycles of the DW theory. The black tetrahedron referred to as a "factor" is the analogue of a bubble that is contracted. The equality is based on absorbing this black tetrahedron and is thus the analogue of separability in 2+1 dimensional topological order. In the current model however the factor is equal to unity.

Search for critical point between electric and magnetic boundaries:

Using the same method — we can find the critical temperature of 2+1 D ising model as a phase transition between two of the three Higher Frobenius algebra of the 4D toric code.
— to appear soon

3D Ising: bond=1, transition temperature: 0.27-0.28

, various values of D. For a discussion see the tex

D	β_c
Ising	0.22165463(8)
0.641	0.38567122(5)
0.655	0.387721735(25)
$\ln 2 = 0.69314718...$	0.39342239(8)
1.15	0.4756110(2)
1.5	0.5575303(10)

Thank you very much!